Complexified diffeomorphism groups, totally real submanifolds and Kähler-Einstein geometry

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Abstract

Let (M, J) be an almost complex manifold. We show that the infinite-dimensional space $\mathcal{T}$ of totally real submanifolds in M carries a natural connection. This induces a canonical notion of geodesics in $\mathcal{T}$ and a corresponding definition of when a functional $f : \mathcal{T} \to \mathbb{R}$ is convex.

Geodesics in $\mathcal{T}$ can be expressed in terms of families of J-holomorphic curves in M; we prove a uniqueness result and study their existence. When M is Kähler we define a canonical functional on $\mathcal{T}$; it is convex if M has non-positive Ricci curvature.

Our construction is formally analogous to the notion of geodesics and the Mabuchi functional on the space of Kähler potentials, as studied by Donaldson, Fujiki and Semmes. Motivated by this analogy, we discuss possible applications of our theory to the study of minimal Lagrangians in negative Kähler–Einstein manifolds.

1 Introduction

Let (M, J) be a $2n$-dimensional manifold endowed with an almost complex structure. Given $p \in M$, we say an n-plane π in T_pM is *totally real* if $J(\pi) \cap \pi = \{0\}$, *i.e.* if T_pM is the complexification of π. An n-dimensional submanifold L is *totally real* if T_pL is totally real in T_pM for all $p \in L$. This gives a decomposition

$$T_pM = T_pL \oplus J(T_pL).$$

Although totally real submanifolds are a natural object in complex geometry, they cannot be studied using purely complex analytic tools. They are, in a sense, the opposite of complex submanifolds; in fact, they are "maximally non-complex", where maximal also refers to their dimension. Furthermore, the defining condition is an open one so their "moduli space" $\mathcal{T}$ is infinite-dimensional.

It might seem reasonable to conclude that this class of submanifolds is too weak to carry interesting geometry. In this paper we will prove the contrary by initiating a study of the global geometric features of the space $\mathcal{T}$. Further results in this direction appear in the companion paper [11].

Geodesics on $\mathcal{T}$. Our first main result, described in Section 2.1, is that $\mathcal{T}$ admits a natural connection, inducing a notion of geodesics. In simpler language, we discover that there exist canonical 1-parameter deformations of a totally real submanifold L, in any given direction. This is rather striking: there is no analogue of this fact known in other spaces of submanifolds. In some sense this observation is the "global version" of the definition of totally real submanifolds, which says the "normal" space T_pM/T_pL and tangent space T_pL are canonically isomorphic via J. In other words, the extrinsic and intrinsic geometry of L coincide; geodesics are, in a sense, the extrinsic analogue of the integral curves of tangent vector fields.

A convex functional. The geodesics induce a notion of convex functionals $f : \mathcal{T} \to \mathbb{R}$: specifically, those which are convex in one variable when restricted to each geodesic. A second striking fact is provided by the following example. Consider $M = \mathbb{C}$, so that $\mathcal{T}$ is the space of curves: in this situation we prove that the standard length functional is convex in our sense. Interestingly, this turns out to be a reformulation of a classical result due to Riesz concerning certain convexity properties of integrals of the form $r \mapsto \int |u(re^{i\theta})|\, d\theta$, where u is a subharmonic function on an annulus.

The length functional uses the metric on $\mathbb{C}$, so in higher dimensions it is thus natural to focus on Kähler, more generally almost Hermitian, manifolds M and look for an analogous functional on $\mathcal{T}$. A first guess might be to use the standard Riemannian volume functional but, in our context, this functional is rather unnatural because it does not encode the totally real condition. In the literature [1] one finds a second "volume functional", tailored specifically to totally real submanifolds.

To understand this alternative functional, the key observation is that there exists a second, equivalent, definition of the totally real condition: L is totally real if and only if the pullback operation for forms defines an isomorphism $K_{M|L} \simeq \Lambda^n(L;\mathbb{C})$. One can view this as another manifestation of the "extrinsic=intrinsic" property of totally real submanifolds. When L is oriented it turns out that $K_{M|L}$ admits a canonical section. Integrating this (real) n-form on L defines the "J-volume functional", which agrees with the length functional in dimension 1 but is in general different to the Riemannian volume functional.

Our second main result, stated in Section 5.4, is that, in the appropriate setting, this functional is convex in our sense.

Applications to minimal Lagrangian submanifolds. This brings us to another set of results in this paper, describing the relationship between the J-volume and the standard Riemannian volume.

The outcome is particularly interesting when M is a Kähler–Einstein (KE) manifold with negative scalar curvature. Recall that an n-dimensional submanifold L in M is Lagrangian if the ambient Kähler form $\bar{\omega}$ vanishes when restricted to L. Lagrangian submanifolds are a key topic in symplectic geometry. In the Kähler case it is particularly interesting to study interactions between symplec-

tic and Riemannian properties of L. For example, it is well-known that (i) in KE manifolds the mean curvature flow preserves the Lagrangian condition and (ii) in negative KE manifolds the minimal Lagrangians are strictly stable for the standard Riemannian volume.

Fact (i) is the starting point for [11]. Our goal here is to further investigate fact (ii). Specifically, when M is negative KE we show the following.

- The J-volume provides a lower bound for the standard Riemannian volume. The two functionals coincide on Lagrangian submanifolds.
- The critical points of the J-volume are exactly the minimal Lagrangian submanifolds. It thus "weeds out" the additional critical points (non-Lagrangian minimal submanifolds) of the standard Riemannian volume.
- The J-volume is strictly convex with respect to our geodesics. For a minimal Lagrangian this is the global counterpart of the infinitesimal stability property mentioned above.

It is thus clear that the J-volume provides very good control over minimal Lagrangians. No such convexity holds for the Riemannian volume functional.

A moment map. The above results fit into a much larger picture. Indeed, the geometric features of $\mathcal{T}$ resemble those of two other, very well-known, infinite-dimensional spaces which appear in Kähler geometry: the integrable $(0,1)$-connections on a Hermitian vector bundle E, *i.e.* the holomorphic structures on E, and the Kähler potentials in a given Kähler class. In both cases we have the following.

- A canonical connection and notion of geodesics, related to an infinite-dimensional group action and its formal "complexification".
- A convex functional.
- A moment map encoding the group action, whose zero set coincides with the critical point set of the functional.

Following this lead, in Section 8 we show that the geometry of $\mathcal{T}$ can be rephrased in terms of the formal complexification of the group of (orientation-preserving) diffeomorphisms of L and of a moment map induced by the J-volume functional. In particular, in the negative KE context it follows that minimal Lagrangians can be re-interpreted as the zero set of a moment map.

Open problems. The above results lead to some interesting questions concerning minimal Lagrangians and their relationship to the "intrinsic" geometry of negative KE manifolds.

In the analogous problem concerning Kähler potentials, the moment map point of view serves to relate the existence of critical points of the functional to certain algebraic stability properties of the manifold. The uniqueness of these

points is instead related to the convexity of the functional. This formalism provides in particular a useful understanding of the geometry of Fano manifolds, and was indeed one of the ingredients of the recently accomplished existence theory for positive KE metrics.

By contrast, the existence of KE-flat (Calabi–Yau) metrics was solved by Yau in the 1980s. Currently, the main questions in this setting are related to calibrated geometry, mirror symmetry and its applications to String Theory in Physics.

Given that the existence of negative KE metrics was also completely solved in the 1980s, by Aubin and Yau, one might wonder what are the most interesting open questions in this context. Our results provide evidence that minimal Lagrangians are closely related to deep aspects of this geometry. They also show that the J-volume is a useful tool with which to probe this relationship.

On a technical level, the key feature of the space of Kähler potentials was its amenability to analytic methods, ultimately leading (across 20 years) to a complete existence theory for geodesics and to the corresponding extension of convexity results. Analytically, the main question we set up in this paper is whether an analogous theory is possible for geodesics in $\mathcal{T}$. In Section 3 we provide a reformulation of the geodesic equation in terms of families of J-holomorphic curves intersecting the initial submanifold L. In the holomorphic setting this helps elucidate key features of the equation by allowing us to use standard techniques from one complex variable to build examples and counterexamples to the existence of solutions to the geodesic equation. It also provides a fairly complete understanding of the uniqueness problem for geodesics. It is clear however that the final answer to these questions will require substantial effort, on a completely different technical scale.

A second interesting problem concerns the existence and uniqueness of minimal Lagrangian submanifolds in negative KE manifolds. As in the Kähler potential story, this may be related to a stability-type condition on the given data. In particular, one might be interested in studying whether minimal Lagrangians persist under deformations of the ambient KE structure: we refer to the final section of [11] for comments in this direction.

A related question is whether minimal Lagrangians are isolated. In a finite-dimensional context it would suffice for the functional to be strictly convex at its critical points, but in our infinite-dimensional setting we would need much stronger information to reach this conclusion: basically, uniform bounds in the various directions. The convexity of the J-volume functional may also help towards this question.

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2 The space of totally real submanifolds

To start, let us make three initial choices:

- a $2n$-dimensional manifold (M, J) endowed with an almost complex structure;
- an oriented n-dimensional manifold L;
- a totally real immersion $\iota : L \to M$: this serves only to determine the homotopy class of immersions we will be interested in.

It will be important for us to maintain the distinction between immersions of L and their corresponding images, *i.e.* "submanifolds". In general, a *submanifold* is an equivalence class of immersions, up to reparametrization via a diffeomorphism of L. In our case, since the orientation of L will play a role, we will be interested in a slightly more refined notion: an *oriented submanifold* is an equivalence class of immersions, up to reparametrization by orientation-preserving diffeomorphisms.

Notice that the totally real condition is preserved under reparametrization, so it is well-defined on the space of (possibly oriented) submanifolds. We can now define our two main spaces of interest.

- Let $\mathcal{P}$ denote the space of totally real immersions of L into M which are homotopic, through totally real immersions, to the given ι.
- Let $\mathcal{T}$ denote the corresponding space of oriented totally real submanifolds, obtained as the quotient of $\mathcal{P}$ by the group of orientation-preserving diffeomorphisms of L (to simplify notation we will denote this group by $\mathrm{Diff}(L)$, omitting any reference to the orientation).

We shall view the data $\pi : \mathcal{P} \to \mathcal{T}$, where π is the natural projection, as a principal fibre bundle with respect to the right group action of $\mathrm{Diff}(L)$ determined by reparametrization. Observe that the totally real condition is open in the Grassmannian of all tangent n-planes in M, so it is a "soft" condition. In particular, $\mathcal{P}$ is an open subset of the space of all immersions. It thus has a natural Fréchet structure, making it an infinite-dimensional manifold. Given any $\iota \in \mathcal{P}$, we can identify $T_\iota\mathcal{P}$ with the space of all sections of (the pull-back of) the bundle TM over L.

Moreover, $\mathcal{T}$ is (at least formally) also an infinite-dimensional manifold. Given $L \in \mathcal{T}$, its tangent space $T_L\mathcal{T}$ can be obtained via the infinitesimal analogue of the operation which quotients immersions by reparametrization. Specifically, $T_L\mathcal{T}$ can be identified with sections of the bundle $TM/TL \simeq J(TL) \simeq TL$ over L; we conclude that $T_L\mathcal{T} \simeq \Lambda^0(TL)$. The key point is that the totally real condition not only provides a canonical subspace in TM which is transverse to TL, but also a canonical isomorphism of this space with TL. In other words, the (extrinsic) "normal" bundle (defined via quotients) is canonically isomorphic to the (intrinsic) tangent bundle.

Remark The action of $\mathrm{Diff}(L)$ might not be free in general; it is guaranteed to be free only in the case of embeddings. We will not worry about this issue, just as we will not be concerned about the precise definitions of infinite-dimensional manifolds, Lie groups and bundles. Everything concerning such matters is to be taken as purely formal, but it will provide vital insight into the intrinsic geometry of $\mathcal{T}$. We refer to [9] for one approach to infinite-dimensional geometry and analysis which could be applied in our setting.

Remark Some orientable manifolds, *e.g.* the spheres $\mathbb{S}^n$, admit an orientation-reversing diffeomorphism ϕ. In this case, reparametrization by ϕ defines a natural $\mathbb{Z}_2$-action on the space of immersions; two initial choices of totally real immersion related this way define different (non-homotopic) spaces $\mathcal{P}$, thus $\mathcal{T}$. Other orientable manifolds do not admit such diffeomorphisms: *e.g.* $\mathbb{CP}^2$. In this case there is no distinction between submanifolds and oriented submanifolds.

2.1 A canonical connection and geodesics

Differentiating the action of $\mathrm{Diff}(L)$ at $\iota \in \mathcal{P}$ we obtain a subspace V_ι of the tangent bundle $T_\iota\mathcal{P}$, canonically isomorphic to the Lie algebra $\Lambda^0(TL)$ of vector fields on L. The space V_ι is the kernel of $\pi_*[\iota] : T_\iota\mathcal{P} \to T_{\pi(\iota)}\mathcal{T}$ and is given by

$$V_\iota = \{\iota_* X : X \in \Lambda^0(TL)\}.$$

Consider

$$H_\iota := J(V_\iota) = \{J\iota_* X : X \in \Lambda^0(TL)\}.$$

This space gives a complement to V_ι, in the sense that there is a decomposition

$$T_\iota\mathcal{P} = V_\iota \oplus H_\iota.$$

Varying ι in $\mathcal{P}$ we obtain a distribution H in $T\mathcal{P}$.

Let $\varphi \in \mathrm{Diff}(L)$ and let $\iota \in \mathcal{P}$. Let R_φ denote the right action of φ on $\mathcal{P}$, *i.e.* $R_\varphi\iota = \iota \circ \varphi$. We now show that the distribution H is right-invariant.

Lemma 2.1 *Let $\varphi \in \mathrm{Diff}(L)$ and $\iota \in \mathcal{P}$. Then $(R_\varphi)_* H_\iota = H_{R_\varphi\iota}$.*

Proof: Let $J\iota_* X \in H_\iota$. Then $J\iota_* X \in T_\iota\mathcal{P}$, so by definition there exists a curve ι_t in $\mathcal{P}$ with $\iota_0 = \iota$ and $\frac{\mathrm{d}\iota_t}{\mathrm{d}t}|_{t=0} = J\iota_* X$.

Thus we may calculate for $p \in L$:

$$\begin{aligned}(R_\varphi)_* J\iota_*|_p X|_p &= \frac{\mathrm{d}}{\mathrm{d}t}(R_\varphi \circ \iota_t)|_{t=0,p} = \frac{\mathrm{d}}{\mathrm{d}t}(\iota_t \circ \varphi)|_{t=0,p} = \frac{\mathrm{d}\iota_t}{\mathrm{d}t}|_{t=0,\varphi(p)} \\ &= J\iota_*|_{\varphi(p)} X|_{\varphi(p)} = J\iota_*|_{\varphi(p)} \circ \varphi_*|_p \circ \varphi_*^{-1}|_{\varphi(p)} X|_{\varphi(p)} \\ &= J(\iota_* \circ \varphi_*)|_p (\varphi_*^{-1} X)|_p.\end{aligned}$$

Hence $(R_\varphi)_* J\iota_* X = J(R_\varphi\iota)_*(\varphi_*^{-1} X) \in H_{R_\varphi\iota}$. □

According to the general theory of principal fibre bundles, H thus defines a connection on (the principal fibre bundle) $\mathcal{P}$.

Recall also from the general theory that any representation ρ of Diff(L) on a vector space E defines an associated vector bundle $\mathcal{P} \otimes_\rho E$ over $\mathcal{T}$; each fibre of this bundle is isomorphic to E. Such a bundle has an induced connection. Parallel sections of this bundle can be described as follows. Choose a curve of submanifolds L_t in $\mathcal{T}$. Choose a horizontal lift ι_t, *i.e.* a curve in $\mathcal{P}$ satisfying $\pi(\iota_t) = L_t$ and $\frac{\mathrm{d}}{\mathrm{d}t}\iota_t \in H_{\iota_t}$. Choose any ($t$-independent) vector $e \in E$. Then the section $[(\iota_t, e)]$ of $\mathcal{P} \otimes_\rho E$, defined along L_t, is parallel. We can obtain all such parallel sections simply by varying e.

In particular, using the adjoint representation of Diff(L) on its Lie algebra gives the vector bundle $\mathcal{P} \otimes_{\mathrm{ad}} \Lambda^0(TL)$. It is of fundamental importance to us that this bundle is canonically isomorphic to the tangent bundle of $\mathcal{T}$, via

$$\mathcal{P} \otimes_{\mathrm{ad}} \Lambda^0(TL) \simeq T(\mathcal{T}), \quad [\iota, X] \mapsto \pi_*[\iota](J\iota_* X).$$

Remark When M is complex (so J is integrable), we will return to this map in Sections 6.2 and 6.4 from a different point of view, as a consequence of Proposition 6.1.

This isomorphism implies that the connection on $\mathcal{P}$ induces a connection on $T(\mathcal{T})$. We can then describe parallel vector fields on $\mathcal{T}$ as above. Finally, recall that a curve L_t is a geodesic if its tangent vector field $\frac{\mathrm{d}}{\mathrm{d}t}(L_t)$ is parallel. We thus obtain the following characterization of geodesics in $\mathcal{T}$.

Lemma 2.2 *A curve L_t in $\mathcal{T}$ is a geodesic if and only if there exists a curve of immersions ι_t and a fixed vector field X in $\Lambda^0(TL)$ such that $\pi(\iota_t) = L_t$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\iota_t = J\iota_{t*}(X).$$

This implies that $[\iota_{t*}X, J\iota_{t*}X] = 0$, for all t for which L_t is defined.

Proof: The form of the equation proves that ι_t is horizontal, and $X \in \Lambda^0(TL)$ plays the role of $e \in E$ in the general theory.

Assume L_t is a geodesic defined for $t \in (-\epsilon, \epsilon)$. Let $x(s)$ be an integral curve of X on L, defined for some $s \in (a, b)$. Then

$$f(s,t) : (a,b) \times (-\epsilon, \epsilon) \to M, \quad f(s,t) = \iota_t(x(s))$$

is an immersed surface in M and $\iota_{t*}X$, $J\iota_{t*}X$ represent its coordinate vector fields in the s and t directions, respectively. As such, they commute. $\square$

Remark The existence of a canonical connection on the space of totally real submanifolds appears to be a rather remarkable fact. One might wonder why this is not true for the space $\mathcal{S}$ of all submanifolds. Although one can argue that there is a canonical right-invariant horizontal distribution on the space of all immersions $\mathcal{I}$, defined by sections of the normal bundle of any submanifold $L \subseteq M$, there appears to be no way to view $T(\mathcal{S})$ as a vector bundle associated to $\mathcal{I}$, so it does not receive an induced connection. In other words, the group action on $\mathcal{I}$ encodes only intrinsic information, and for a general submanifold one cannot encode the extrinsic geometry of the normal bundle intrinsically.

2.2 Convexity

The notion of geodesic induces a natural definition of convexity for functionals on $\mathcal{T}$.

Definition 2.3 *We say that a functional $F : \mathcal{T} \to \mathbb{R}$ is convex if and only if it restricts to a convex function in one variable along any geodesic in $\mathcal{T}$.*

In the absence of existence results for geodesics, this notion could be vacuous. However, in the presence of geodesics, a convex functional provides a powerful tool for analysing the geometry of $\mathcal{T}$. We thus now turn to the existence problem.

3 The geodesic equation

In general, given a notion of geodesics on a manifold, there are two key existence issues which arise: (i) the Cauchy problem, *i.e.* the short-time existence of geodesics given an initial point and an initial direction, and (ii) the boundary value problem, *i.e.* the existence of geodesics between two points in the manifold.

When the manifold is finite-dimensional, or infinite-dimensional and Banach, the first problem is purely local and can be solved via the standard existence theory for ordinary differential equations. The second problem concerns the global properties of the manifold and leads to the notion of geodesic completeness.

In our case the manifold $\mathcal{T}$ is infinite-dimensional but only Fréchet, so existence and uniqueness results for geodesics are non-trivial. The goal of this section is to rephrase our notion of geodesics in terms of families of J-holomorphic curves in (M, J). This has several advantages.

- It offers a geometrically appealing reformulation of the geodesic equation.
- It clarifies the nature of the geodesic equation, indicating for example that it is not elliptic; however, it can be written as a family of elliptic equations.
- It opens the door to standard tools in the theory of one complex variable.

This point of view will lead us, at least when M is complex, to a complete solution of the uniqueness question. It does not give a complete answer to the existence problem, but it does yield useful insight by providing both examples and counterexamples and by suggesting a slight weakening of the notion of solution.

3.1 A reformulation of the geodesic equation

We will distinguish three cases: the Cauchy problem, geodesic rays and the boundary value problem.

The Cauchy problem. Assume we are given an initial submanifold $L_0 \in \mathcal{T}$ and an initial direction in $T_{L_0}\mathcal{T}$, which can be identified with a smooth vector field X on the abstract manifold L. Ideally, the initial value problem for the geodesic equation can then be solved as follows.

1. Choose an initial parametrization ι_0 of the submanifold L_0.

2. Consider the flow defined by X on L; choose an integral curve $x = x(s)$ of X, where $s \in I := (a, b)$.

3. Look for a J-holomorphic curve $\iota(s,t) : I \times (-\epsilon, \epsilon) \to (M, J)$ such that $\iota(s, 0) = (\iota_0 \circ x)(s)$.

   Here, $I \times (-\epsilon, \epsilon)$ is endowed with its standard complex structure.

4. By varying the choice of integral curve we obtain a family of J-holomorphic curves. Since each point of L belongs to some integral curve, by fixing the time parameter t this family defines a map $\iota_t : L \to M$ which coincides with ι_0 for $t = 0$.

If the J-holomorphic curves depend smoothly on the choice of integral curve, ι_t will be smooth. Since the space of immersions is open in the space of maps, ι_t will be an immersion for small t; their image submanifolds in $\mathcal{T}$ solve the geodesic equation by construction.

Though intuitively appealing, this procedure clearly entails some difficulties. In particular, we observe the following.

- Fix X. In order to obtain a map defined on all L for each $t \in (-\epsilon, \epsilon)$ it is necessary to be able to choose ϵ to be independent of the integral curve.

- Question: given a constant C, does there exist $\epsilon > 0$ such that geodesics exist for any X satisfying $\|X\| < C$?

Above, $\|\cdot\|$ could refer for example to the C^0 norm on vector fields, for some metric on L. A positive answer to this question would lead to a well-defined "exponential map" from the ball of radius C in $T_{L_0}\mathcal{T}$ into $\mathcal{T}$.

To proceed further we must determine the correct framework within which to analyze our J-holomorphic equations. We are trying to solve an elliptic problem on the domain $I \times (-\epsilon, \epsilon)$ by prescribing data inside the domain rather than, say, on the boundary. Notice that the domain itself is not prescribed: ϵ is to be determined. It is thus natural to try to solve this problem using the "method of characteristics". The initial data is assigned on the curve $I \times \{0\} \subset I \times (-\epsilon, \epsilon)$: this curve is non-characteristic for our equation, so the method makes sense.

In this setting the only general existence result is the Cauchy–Kowalevski theorem, which applies only to real analytic initial data. In many ways this regularity restriction is strong: from the geometric viewpoint it is natural to solve the Cauchy problem for geodesics in the space of smooth immersions, built as above using maps $C^\infty\big(I \times (-\epsilon, \epsilon), M\big)$. On the other hand, when M is complex, standard regularity theory implies that any solution is automatically

complex analytic with respect to $s + it$. In particular this implies that, if the solution exists, the initial data $\iota_0 \circ x(s)$ must be real analytic. We conclude that the analytic setting is actually a very natural one for the geodesic problem, as stated above. Within this context we will see below that the Cauchy–Kowalevski theorem provides very strong existence results.

This same reasoning also indicates however that the regularity theory yields a fundamental obstruction to the existence of solutions to the above Cauchy problem when the initial data is only assumed to be smooth. It is thus important to introduce a weaker notion of geodesic, as follows.

Geodesic rays. In the standard theory of one complex variable it is natural to study maps defined on closed domains: holomorphic on the interior, but only smooth or continuous up to the boundary. This leads us to the following.

Definition 3.1 *Fix $L_0 \in \mathcal{T}$ and $JX \in T_{L_0}\mathcal{T}$. A geodesic ray starting from L_0 with direction JX is a curve of submanifolds L_t in $\mathcal{T}$, for $t \in (0, \epsilon)$, for which there exists a curve of immersions ι_t, for $t \in [0, \epsilon)$, with the following properties:*

- *ι_t is smooth on $L \times [0, \epsilon)$;*
- *for $t \in (0, \epsilon)$, ι_t solves the equation $\frac{d}{dt}\iota_t = J\iota_{t*}(X)$;*
- *ι_0 parametrizes L_0.*

The problem of existence of geodesic rays is manifestly different to the Cauchy problem previously described.

The boundary value problem. Using the above ideas, we can now define geodesics between two submanifolds L_0 and L_1 in $\mathcal{T}$ as geodesic rays interpolating between them.

To prove the existence of such a geodesic it is necessary to find a vector field X on L and smooth, totally real immersions $\iota_t : L \to M$, for $t \in [0, 1]$, so that:

- $(x, t) \mapsto \iota_t(x)$ is smooth on $L \times [0, 1]$;
- for $t \in (0, 1)$, $\frac{d}{dt}\iota_t = J\iota_{t*}(X)$;
- ι_0, ι_1 parametrize L_0, L_1.

As above, we can decompose a geodesic ray into a family of J-holomorphic curves parametrized by the set of integral curves of X on L, thus defined on domains of the sort $I \times [0, \epsilon)$. For the boundary value problem, each such curve provides a J-holomorphic filling between the boundary data prescribed by ι_0 on $I \times \{0\}$ and ι_1 on $I \times \{1\}$.

Remark If X defines a geodesic ray for $t \in [0, \epsilon)$ then $-X$ defines a geodesic ray for $t \in (-\epsilon, 0]$ and the two induced families L_t coincide, up to time reversal.

3.2 Existence in the real analytic case

To study the existence of geodesics in the real analytic category we begin by stating the Cauchy–Kowalevski theorem (see e.g. [14, Chapter 10 Theorem 4]).

Theorem 3.2 (Cauchy–Kowalevski) *Let $U \subseteq \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{mk}$ be open and let $G : U \to \mathbb{R}^k$ be real analytic. Let $U_0 \subseteq \mathbb{R}^m$ be open and let $f : U_0 \to \mathbb{R}^k$ be real analytic such that $\{(x, t_0, f(x), Df(x)) : x \in U_0\} \subseteq U$ for some $t_0 \in \mathbb{R}$.*

Then (possibly restricting U_0) there exists an open neighborhood $U_1 \subseteq \mathbb{R}^m \times \mathbb{R}$ of $U_0 \times \{t_0\}$ and a real analytic map $F : U_1 \to \mathbb{R}^k$ satisfying the partial differential equation with initial condition

$$\frac{\partial F}{\partial t} = G\left(x, t, F, \frac{\partial F}{\partial x}\right) \quad \text{and} \quad F(x, t_0) = f(x) \text{ for all } x \in U_0.$$

Moreover, F is unique in the sense that any other real-analytic solution of this equation and initial condition agrees with F on some neighborhood of $U_0 \times \{t_0\}$.

Suppose L and M are real analytic, L is compact and $\iota_0 : L \to M$ is a real analytic totally real immersion. Let $L_0 = \iota_0(L)$ and consider elements in $T_{L_0}\mathcal{T}$ which correspond to real analytic vector fields X on L.

Given $p \in L$ we can find a neighbourhood $V_p \subseteq L$ of p in L, $\rho_p > 0$ and $\delta_p > 0$ such that for all $q \in V_p$ we have a unique integral curve $\alpha_q : (-\delta_p, \delta_p) \to L$ of X passing through q whenever $\|X\|_{C^0} \leq \rho_p$, *i.e.* $\alpha_q'(s) = X(\alpha_q(s))$ and $\alpha_q(0) = q$. The existence of V_p, ρ_p and δ_p follow from the theory of first order ordinary differential equations and the dependence of the solution on the initial data. (In particular, to show it suffices to have the same V_p and δ_p for any X with a C^0 bound depending only on p, one may employ the "method of majorants", discussed below.) It is also a consequence of this theory that, when X is real analytic, α_q is real analytic for all q. By making V_p smaller if necessary we can assume it is contained in a chart on L and hence may be identified with a subset of $\mathbb{R}^n$.

We then have an initial real analytic map $f_p : V_p \times (-\delta_p, \delta_p) \to M$ given by

$$f_p(q, s) = \iota_0(\alpha_q(s)).$$

We may assume by making V_p, δ_p and ρ_p smaller that the image of f is contained in a chart W_p on M which may be identified with an open subset of $\mathbb{R}^{2n}$, so that f_p maps an open set in $\mathbb{R}^{n+1}$ into $\mathbb{R}^{2n}$. We then wish to find $\epsilon_p > 0$ and $F_p : V_p \times (-\delta_p, \delta_p) \times (-\epsilon_p, \epsilon_p) \to M$ satisfying

$$\frac{\partial F_p}{\partial t}(q, s, t) = J_{F_p(q,s,t)} \frac{\partial F_p}{\partial s}(q, s, t).$$

Notice here the role of q: it keeps track of the specific integral curve we are working with, ultimately ensuring uniform behaviour also with respect to this choice.

If J is also assumed to be real analytic, then the right-hand side can be viewed as $G(q,s,t,F_p,\frac{\partial F_p}{\partial s})$ for a real analytic function G defined on a neighbourhood of $V_p\times(-\delta_p,\delta_p)\times\{0\}\times W_p\times Z_p$, where Z_p is an open set in $\mathbb{R}^{2n(n+1)}$ containing Df_p.

Hence we may apply the Cauchy–Kowalevski Theorem (Theorem 3.2) and deduce that the initial data f_p determines a real analytic solution F_p and constant $\epsilon_p>0$. Moreover, the uniqueness part of the theorem means that we can assume F_p and ϵ_p to be maximal in the sense that if there is another real analytic solution on $V_p\times(-\delta_p,\delta_p)\times(-\epsilon_p,\epsilon_p)$ then it must equal F_p.

Now, although the equation for F_p is independent of X, the initial condition f_p is determined by X so the "existence time" ϵ_p of the solution F_p also depends on X. We claim, however, that ϵ_p can be chosen to only depend on the bound ρ_p on X. To see this, one must inspect the proof of Theorem 3.2 (see e.g. [14, Chapter 10 Section 4]).

One of the ideas in the proof is to change the initial condition to 0 by modifying the equation so that it depends on the initial condition. Since the problem is first order, this modification leads to new coefficients in the equation which depend on the first derivatives of the initial condition. In our setting, this amounts to changing f_p to $(q,s)\mapsto\iota_0(q)$ (so we are choosing $X=0$), and then changing the operator G so that it depends on the derivative of $f_p(q,s)=\iota_0(\alpha_q(s))$, which means pointwise on X. We thus convert the problem into solving $\frac{\partial F_p}{\partial t}=G_X(F_p,\frac{\partial F_p}{\partial s})$ with $F_p(q,s,0)=\iota_0(q)$.

Another key idea is to use the "method of majorants". This amounts to stuying the series expansion of the modified operator: some coefficients are "universal", others depend on the specific original operator and on the initial conditions. It turns out that the latter coefficients can be bounded in terms of a C^0-bound on the initial data: the resulting equation can be solved real analytically on a certain domain U_1, and one can check that the original equation also admits a solution on that same domain. In our case, by assuming a sufficiently small bound $\|X\|_{C^0}\leq\rho_p$ we can thus ensure that the existence time ϵ_p only depends on ρ_p.

For all $p\in L$ and X real analytic with $\|X\|_{C^0}\leq\rho_p$, we therefore have neighbourhoods V_p of p in L, positive constants δ_p and ϵ_p and real analytic maps $F_p:V_p\times(-\delta_p,\delta_p)\times(-\epsilon_p,\epsilon_p)\to M$ such that $\frac{\partial F_p}{\partial t}=J\frac{\partial F_p}{\partial s}$ and $s\mapsto F_p(q,s,0)$ is the integral curve of X through q for all $q\in V_p$.

Since L is compact, it is covered by finitely many such neighbourhoods, say $V_{p_1},\ldots,V_{p_k}$, and hence the integral curves of any real analytic vector fields X satisfying $\|X\|_{C^0}\leq\rho:=\min_j\{\rho_{p_j}\}>0$ are defined for $s\in(-\delta,\delta)$ where $\delta=\min_j\{\delta_{p_j}\}>0$. We also have real analytic maps F_{p_j} defined on $V_{p_j}\times(-\delta,\delta)\times(-\epsilon,\epsilon)$ where $\epsilon=\min_j\{\epsilon_{p_j}\}$. Moreover, the maximality condition on the F_{p_j} means that they agree on any overlaps of the finite cover of L. We thus obtain a well-defined real analytic map $F:L\times(-\delta,\delta)\times(-\epsilon,\epsilon)\to M$ so that $(s,t)\mapsto F(p,s,t)$ defines a J-holomorphic curve extending the integral curve of X through p. We therefore may define $\iota_t:L\to M$ for $t\in(-\epsilon,\epsilon)$ by $\iota_t(p)=F(p,0,t)$: this is an immersion for δ sufficiently small and the family ι_t

defines a geodesic.

These observations prove the existence of an "exponential map" for $\mathcal{T}$ in the real analytic context, in the following sense.

Theorem 3.3 *Let L be a compact real analytic n-manifold, let (M, J) be a real analytic almost complex $2n$-manifold such that J is also real analytic, and let $\iota_0 : L \to M$ be a real analytic immersion.*

There exist $\rho > 0$ and $\epsilon > 0$ such that, if X is a real analytic vector field on L with $\|X\|_{C^0} \leq \rho$, then there exists a geodesic $(L_t)_{t \in (-\epsilon, \epsilon)}$ in $\mathcal{T}$ with $L_0 = \iota_0(L)$ given by totally real immersions $\iota_t : L \to M$ with $\frac{d}{dt}\iota_t = J\iota_{t*}(X)$ and $\iota_t|_{t=0} = \iota_0$.

3.3 Example: the 1-dimensional case

We now want to turn to smooth data. It is instructive to study the Cauchy problem in the simplest possible case, in which $M = \mathbb{C}$ and L_0 is a smooth, closed, Jordan curve. Recall that the complement of such a curve has two components: one bounded, one unbounded. We will view L_0 as an embedding $\iota_0 = \iota_0(\theta)$ of the abstract manifold $L := \mathbb{R}/2\pi\mathbb{Z}$. For dimensional reasons any such embedding is totally real. For the sake of being concrete we will use the standard orientation on L defined by increasing angles and we will assume that the embedding is chosen so that L_0 is oriented in the anti-clockwise direction.

According to our definition, geodesics through L_0 are generated by a choice of tangent vector field. Since L is parallelizable and has the canonical, positively oriented, vector field $\partial\theta$, any tangent field X can be written as $f\partial\theta$, for some $f : L \to \mathbb{R}$. The corresponding geodesic in $\mathcal{T}$ is determined by the images of the 1-parameter family of curves

$$\iota : L \times (-\epsilon, \epsilon) \to \mathbb{C}$$

such that $\iota = \iota_0$ for $t = 0$ and

$$\frac{\partial \iota}{\partial t} = if\frac{\partial \iota}{\partial \theta}. \tag{1}$$

This coincides with the geodesic equation as it appears in Lemma 2.2. In particular, when $f \equiv 1$ this means that ι is holomorphic with respect to the standard complex structure on the cylinder $L \times (-\epsilon, \epsilon)$. We can use the biholomorphism with the annulus

$$\phi : L \times (-\epsilon, \epsilon) \to A := \{e^{-\epsilon} < |z| < e^{\epsilon}\}, \quad \phi(\theta, t) := e^{-t}e^{i\theta} \tag{2}$$

to reparametrize ι as a holomorphic map $g := \iota \circ \phi^{-1} : A \to \mathbb{C}$. Our choice of orientations imply that, as the parameter θ increases, its image will travel along L_0 in the anti-clockwise direction. This implies that, as t increases from 0, the geodesics invade the bounded component of the complement of L_0 in $\mathbb{C}$.

We now want to show that the $f \equiv 1$ example is, in some sense, extremely general. Indeed, using the ideas of Section 3.1, we can alternatively decide to integrate the field $f\partial\theta$. If f has no zeros, *i.e.* the vector field never vanishes, then

the integral curve through any given point of L is periodic and its parameter set is compact: we can identify it with $\mathcal{S}^1_R := \mathbb{R}/2\pi R\mathbb{Z}$, for some $R > 0$. It follows that the integral curve is a (possibly orientation-reversing) diffeomorphism

$$\mathcal{S}^1_R \to L, \quad s \mapsto \theta(s) \ \text{ such that } \ \theta' = f \circ \theta. \tag{3}$$

The composed map $\iota(\theta(s), t)$ is then holomorphic on the cylinder $\mathcal{S}^1_R \times (-\epsilon, \epsilon)$ endowed with the standard complex structure. Again, we can use the biholomorphism with the annulus

$$\phi : \mathcal{S}^1_R \times (-\epsilon, \epsilon) \to A := \{e^{-\epsilon/R} < |z| < e^{\epsilon/R}\}, \quad \phi(\theta, t) := e^{-t/R} e^{i\theta/R} \tag{4}$$

to reparametrize ι as a holomorphic map $g : A \to \mathbb{C}$.

We summarize this discussion as follows.

Lemma 3.4 *Assume we are given a geodesic family of curves L_t defined by L_0 and a nowhere-vanishing vector field $X = f\partial\theta$. The family L_t can then be parametrized via a holomorphic map g defined on an annulus in $\mathbb{C}$ containing $\mathbb{S}^1$. Each L_t is the image under g of some circle $\{|z| = r\}$; in particular L_0 is the image of $\mathbb{S}^1$ and X corresponds to $\pm\partial\theta$ depending on the sign of f.*

Assume for example that f is positive. Then, as the radial parameter r decreases from 1, the corresponding curves invade the bounded component of the complement of L_0 in $\mathbb{C}$.

Remark The above discussion indicates that the 1-dimensional case has a special feature. Recall from Lemma 2.1 that the horizontal distribution H on $\mathcal{P}$ is invariant under reparametrization. Geometrically, this means that we can find all geodesics through L_0 by fixing any initial parametrization ι_0 and considering all possible vector fields: the geodesic in $\mathcal{T}$ defined by a different choice $(\iota_0 \circ \phi, X)$ will coincide with the geodesic defined by $(\iota_0, \phi_* X)$. Thus, in general there is no advantage to changing the parametrization. In dimension 1, however, $\text{Diff}(L)$ acts transitively on the space of non-vanishing vector fields (up to a change of scale). Above, we use this fact to bring X into "standard form" $\partial\theta$, thus to reduce the geodesic equation to the standard Cauchy–Riemann equation. We should however note that, when $f < 0$, this strategy clashes with our initial decision to work with oriented submanifolds, *i.e.* to only use orientation-preserving diffeomorphisms. On the other hand, this is easily fixed by the observation that the geodesic defined by $-X$ coincides with the geodesic defined by X, up to time reversal. To find all geodesics, it it thus enough to concentrate on those for which $f > 0$. A similar remark applies to vector fields with zeros (see below).

If the vector field $X = f\partial\theta$ has zeros, then between any two zeros of f the new parameter set will be $\mathbb{R}$ and the geodesic equation pulls back to the standard holomorphic equation on $\mathbb{R} \times (\epsilon, \epsilon)$. The zeros themselves correspond to stationary points of the family of curves.

As already mentioned, there is a necessary condition for the existence of solutions to this equation: the initial curve must be real analytic. This condition

is also sufficient: given a local power series expansion of ι_0 with respect to the real variable θ, we obtain a holomorphic extension by replacing θ with $\theta + it$.

In Section 3.1, in order to try to remain in the smooth category, we introduced the notion of geodesic ray. Using these same ideas we can investigate this notion in the 1-dimensional case, obtaining the following conclusion.

Lemma 3.5 *Assume we have a geodesic ray L_t defined by L_0 and a nowhere-vanishing vector field X. The family L_t can then be parametrized via a holomorphic map g defined on an annulus in $\mathbb{C}$ of the form $R_1 < |z| < 1$. According to the definition of geodesic ray, g is smooth up to the boundary component $\{|z| = 1\}$. Each L_t is the image under g of some circle $\{|z| = r\}$; in particular L_0 is the image of $\mathbb{S}^1$ and X corresponds to $\pm\partial\theta$ depending on the sign of f.*

Assume for example that f is positive. Then, as the radial parameter r decreases from 1, the corresponding curves invade the bounded component of the complement of L_0 in $\mathbb{C}$.

Elliptic regularity theory shows that, if the boundary data is smooth, the map g will be smooth up to $|z| = 1$ even if in Definition 3.1 we had assumed that the geodesic ray were only continuous with respect to the variable t, at $t = 0$.

Up to now we have been assuming that the geodesics were already given, so let us now turn to the existence question.

Facts from Fourier theory. As a first step let us rephrase our results above in terms of power series. Recall that to any convergent complex power series $\sum_{n=0}^{\infty} a_n z^n$ one can associate the *radius of convergence* $R > 0$ such that the corresponding disk of radius R is the maximal open disk on which the series converges; the limit is a holomorphic function $g : \{|z| < R\} \to \mathbb{C}$.

The series may also converge at some points on the boundary of that disk. For simplicity, let us take $R = 1$. If we assume the series converges at a boundary point $e^{i\theta_0}$, then a theorem of Abel proves radial continuity up to $e^{i\theta_0}$:

$$\sum_{n=0}^{\infty} a_n e^{in\theta_0} = \gamma \quad \text{implies} \quad \sum_{n=0}^{\infty} a_n r^n e^{in\theta_0} \to \gamma \quad \text{as } r \to 1^-.$$

More generally, one can show convergence to γ within any "Stolz region" based at $e^{i\theta_0}$.

Furthermore, assume the power series converges to a continuous function at every point of the boundary:

$$\sum_{n=0}^{\infty} a_n e^{in\theta} = \gamma(\theta),$$

where $\gamma : \mathbb{S}^1 \to \mathbb{C}$ is continuous. Then one can prove that

$$\sum_{n=0}^{\infty} a_n r^n e^{in\theta} = \sum_{n=0}^{\infty} a_n z^n \to \gamma(\theta) \quad \text{uniformly in } \theta \text{ as } |z| \to 1^-,$$

so the limit function $g : \{|z| \leq 1\} \to \mathbb{C}$ is continuous and coincides with γ on $\mathbb{S}^1$. Elliptic regularity implies that if γ is smooth, then g is also smooth.

Analogously, the change of coordinates $z \mapsto 1/z$ shows that complex power series with negative powers $\sum_{n=-1}^{-\infty} a_n z^n$ converge on external domains of the form $|z| > R$, with radial convergence up to any point of the boundary where the series converges.

It is a standard result of complex analysis that any holomorphic function $g = g(z)$ on an annulus $R_1 < |z| < R_2$, *e.g.* any geodesic obtained via Lemma 3.4, admits a representation via a "Laurent series"

$$g(z) = \sum_{n=-1}^{-\infty} a_n z^n + \sum_{n=0}^{\infty} a_n z^n,$$

where by definition both terms on the right hand side are convergent series: the first converges for $|z| > R_1$, the second for $|z| < R_2$.

The result applies also to geodesic rays: maps as in Lemma 3.5 correspond to Laurent series such that the first term converges for $|z| > R_1$ and the second converges for $|z| < 1$ as well as on the boundary $|z| = 1$. The value of the map on this boundary is the sum of the values of the two terms. The first term is smooth on the boundary. Abel's theorem implies radial convergence of the second term as $r \to 1^-$, thus continuity of the sum up to $r = 1$. Once again, elliptic regularity theory proves smoothness if the boundary data is smooth.

We now turn to Fourier theory. Recall that to any L^2 curve $\gamma : \mathbb{S}^1 \to \mathbb{C}$ one can associate Fourier coefficients a_n, for $n \in \mathbb{Z}$, as follows:

$$a_n := \frac{1}{2\pi} \int_0^{2\pi} \gamma(\theta) \cdot e^{-in\theta}\, d\theta.$$

The coefficients a_n are square-summable; conversely, any such collection of coefficients a_n will define a L^2 curve.

In general, even if the curve is continuous its Fourier series does not converge pointwise: in particular, we cannot recover the curve in this way. This is usually indicated via the notation

$$\gamma \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

reserving "=" for the case in which pointwise convergence does hold.

Pointwise convergence can be ensured via higher regularity of γ or via stronger assumptions on a_n. For example, if a Fourier series is absolutely convergent, *i.e.* $\sum_{n=-\infty}^{\infty} |a_n| < \infty$, then we can split it into a sum of two series

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta} = \sum_{n=0}^{\infty} a_n e^{in\theta} + \sum_{n=1}^{\infty} a_{-n} e^{-in\theta},$$

whose partial sums converge uniformly to a continuous function γ.

Alternatively, the values of γ can be recovered via a different notion of convergence, as follows.

Proposition 3.6 *Consider a curve $\gamma : \mathbb{S}^1 \to \mathbb{C}$. Let a_n denote its Fourier coefficients. If γ is continuous then the corresponding Fourier series $\sum a_n e^{in\theta}$ is Abel summable, i.e. the following holds.*

- *For each $r \in [0,1)$ and $\theta \in [0, 2\pi]$, the limit*
$$\lim_{N\to\infty} \sum_{n=-N}^{N} a_n r^{|n|} e^{in\theta}$$
exists, thus defining a continuous function $g : \{|z| < 1\} \to \mathbb{C}$.
- *We have $g(re^{i\theta}) \to \gamma(e^{i\theta})$ as $r \to 1^-$, uniformly with respect to θ, so g extends to a continuous function $g : \{|z| \leq 1\} \to \mathbb{C}$ which coincides with γ when $|z| = 1$.*

If γ is C^1 then its Fourier series is absolutely convergent. It follows that

$$\lim_{N\to\infty} \sum_{n=-N}^{N} a_n r^{|n|} e^{in\theta} = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} a_{-n} r^n e^{-in\theta} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \bar{z}^n$$

and the partial sums converge uniformly to γ.

Geodesics via Fourier theory. We can combine the power series representation of geodesics and Fourier theory by noticing that the coefficients of the power series must coincide, by continuity, with the Fourier coefficients of the initial curve. We emphasize that these representations depend on the choice of immersions ι_t, not just on the family of curves L_t: more precisely, the power series concerns the map g which was defined by a choice of vector field X, and the Fourier coefficients correspond to the parametrization $\gamma(\theta) := g(e^{i\theta})$ of L_0 which identifies X with $\partial\theta$.

We can thus characterize the existence of such a geodesic curve through L_0 with direction JX entirely in terms of γ, as follows.

Proposition 3.7 *Let $\gamma : \mathbb{S}^1 \to \mathbb{C}$ be a smooth map parametrizing L_0 and let a_n, for $n \in \mathbb{Z}$, denote the Fourier coefficients of γ. There exists a holomorphic map $g : A \to \mathbb{C}$ as in Lemma 3.4, thus a geodesic corresponding to the direction $\partial\theta$, if and only if the Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ converges on A.*

Proof: Assume g exists. Then g coincides with γ on $\mathbb{S}^1$ so the coefficients of the Laurent expansion of g must coincide with a_n. Thus $\sum_{n=-\infty}^{\infty} a_n z^n$ coincides with the Laurent expansion of g, so it converges on A.

Conversely, assume $\sum_{n=-\infty}^{\infty} a_n z^n$ converges on A. Then it defines g with the required properties. $\square$

An analogous result hold for geodesic rays. We can use this to build examples and counterexamples of curves admitting geodesics and geodesic rays in certain directions. We start by choosing coefficients a_n, for $n \in \mathbb{Z}$.

- If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R_2 > 1$ and $\sum_{n=-1}^{-\infty} a_n z^n$ has radius of convergence $R_1 < 1$ then the Laurent series determined by their sum converges on the annulus $R_1 < |z| < R_2$ and defines an embedding γ of $\mathbb{S}^1$. The image curve L_0 admits a geodesic corresponding to $\partial\theta$.

- If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1 and converges up to the boundary and $\sum_{n=-1}^{-\infty} a_n z^n$ has radius of convergence $R_1 < 1$ then the Laurent series determined by their sum converges on the semi-annulus $R_1 < |z| \leq 1$ and defines an embedding γ of $\mathbb{S}^1$. The image curve L_0 admits a geodesic ray corresponding to $\partial\theta$.

- If both series have radius of convergence 1 and converge up to the boundary then the Laurent series degenerates: it converges only on $\mathbb{S}^1$, defining an image curve L_0 which does not admit a geodesic or geodesic ray corresponding to $\partial\theta$.

Remark A similar result to Proposition 3.7 also holds if the initial parametrization $\gamma : \mathbb{S}^1 \to \mathbb{C}$ is only continuous, but the proof is slightly more complicated. Let us examine for example the case of geodesic rays. Assume that the Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ generated by the coefficients of γ converges on an annulus $R_1 < |z| < 1$. Then $\sum_{n=-1}^{-\infty} a_n z^n = \sum_{n=1}^{\infty} a_{-n} z^{-n}$ converges for $|z| > R_1$. On this domain one can prove absolute convergence, so $\sum_{n=1}^{\infty} a_{-n} \bar{z}^{-n}$ also converges there. We conclude that $\sum_{n=1}^{\infty} a_{-n} \bar{z}^n$ converges for $|z| < 1/R_1$. In particular, it converges pointwise on $\mathbb{S}^1$.

Subtracting this from the Abel sum of Proposition 3.6 we deduce that the Abel sum splits into the sum of two convergent power series, *i.e.* the Abel sum coincides with the Laurent series generated by γ. Since the Abel sum converges to γ, we conclude that $\sum_{n=-1}^{-\infty} a_n z^n + \sum_{n=0}^{\infty} a_n z^n$ converges to γ for $|z| = 1$, and thus defines a geodesic ray.

Conversely, if there exists a geodesic ray g then its Laurent expansion converges to γ for $|z| = 1$, so its coefficients coincide with those of γ. We conclude that the Laurent series generated by γ converges on the annulus.

Example 3.8 Set $a_n := 1/n^{\log n}$. Then $\sum_{n=1}^{\infty} a_n z^n$ has radius of convergence 1 and converges absolutely for $|z| \leq 1$, together with all derivatives. We can combine this example with any series $\sum_{n=-1}^{-\infty} a_n z^n$ which converges for $|z| > R_2$ to obtain examples of smooth curves which admit geodesic rays but not geodesics corresponding to the direction $\partial\theta$. We can also combine it with the series $\sum_{n=-1}^{-\infty} |n|^{-\log |n|} z^n$ to obtain a smooth curve which admits neither a geodesic nor a geodesic ray corresponding to that direction.

To obtain examples which are only continuous, set $a_n := 1/n^2$.

Geodesics via the Riemann Mapping Theorem. Choose two closed Jordan curves L_0, L_1 in $\mathbb{C}$ which do not intersect. Let Ω be the region contained between these curves. A version of the Riemann mapping theorem, cf. [3, Theorem 5.8], proves that there exists an annulus A and a biholomorphism $g : A \to \Omega$

which extends continuously to the boundary; if L_0, L_1 are smooth then the biholomorphism extends smoothly to the boundary. The restriction to the boundary provides parametrizations of L_0, L_1; setting $X = \partial\theta$ the theorem shows that for any two curves as above it is possible to solve the boundary value problem.

Remark Notice the regularizing behaviour of the geodesic equation even for the boundary value problem: for all intermediate times $t \in (0, 1)$, the corresponding curves are real analytic.

Concluding remarks. It may be useful to summarize what we have learned from the 1-dimensional theory. Given an embedded curve $L_0 \subset \mathbb{C}$, we have shown the following.

- Infinitesimal deformations correspond to parametrizations (via integration of the tangential vector field $f\partial\theta$).
- A geodesic in a given direction corresponds to a holomorphic extension of the corresponding parametrization.
- Examples show that certain curves do not admit geodesics in certain directions.
- Given any curve L_0, there always exist infinite geodesic rays departing from it (corresponding to the arbitrary choice of a second curve L_1).

This suggests that the existence question for geodesics is non-trivial, but also not vacuous. A similar situation occurs in the analogous theory concerning Kähler metrics, cf. Section 7. There a weak notion of geodesics was found, leading to a completely satisfactory existence theory. We expect that something similar is needed here. In particular, observe that our geodesic equation is of first order, rather than the second order one might have expected: this corresponds to the fact that, in keeping with the principal fibre bundle point of view, it is expressed in terms of the velocity vector (being constant) rather than of the curve. Developing alternative expressions for geodesics and further investigation of the properties of the connection may contribute key ingredients to the existence theory. This is work in progress.

3.4 Further results

Some of these same ideas can be extended to higher dimensions.

Existence and non-existence results when $M = \mathbb{C}^n$. Consider a compact totally real submanifold L_0 in $\mathbb{C}^n$ and a tangent vector field X. Choose an integral curve $x = x(s)$ and a parametrization ι_0, with components ι_0^i. If the curve is closed we can study the existence of holomorphic extensions of $\gamma := \iota_0 \circ x(s)$ exactly as when $n = 1$, simply examining its component functions $\iota_0^i \circ x(s)$. This does not work if the curve is open, parametrized by $\mathbb{R}$. Notice however that the image of γ is contained in L_0, so it is bounded. We

can thus interpret γ as a (smooth) tempered distribution and replace the role of Fourier coefficients with Fourier transforms. In particular, we can expect to obtain information concerning the existence of holomorphic extensions of γ using the Paley–Wiener theorems. It is known for example, cf. [13, Theorem 7.23], that if the transform of γ has compact support then γ admits an entire holomorphic extension (satisfying certain growth conditions). Notice that in this case the transform of γ will generally not be smooth, otherwise it would be L^2 so γ would also be L^2.

This applies also to any complex manifold M, as long as the submanifold is contained in one chart.

Uniqueness of geodesics. Perhaps the most interesting feature of our reformulation of the geodesic equation is that it gives a fairly complete answer to the uniqueness question. Indeed, by restricting ι_0 to each integral curve we see that it suffices to prove the following claim: any two J-holomorphic maps $\iota(s,t)$, $\iota'(s,t)$ which coincide for $t = 0$ coincide for all t.

In the holomorphic case (when J is integrable) the proof is simple. As above, ι corresponds locally to a collection of holomorphic functions, defined by its components in $\mathbb{C}^n$. Uniqueness for the Cauchy problem then follows from the standard identity principle for holomorphic functions. Uniqueness for geodesic rays follows instead from the standard reflection principle.

If J is only almost complex the situation is more subtle. Uniqueness for the Cauchy problem is then a consequence of the "unique continuation theorem" for J-holomorphic curves, cf. [12, Theorem 2.3.2]. It seems reasonable that, using results in the literature, one could also prove uniqueness for geodesic rays.

Remark In the real analytic case the uniqueness of real analytic solutions is of course part of the Cauchy–Kowalevski theorem. One might hope to improve on this, obtaining uniqueness in the smooth category, using Holmgren's uniqueness theorem, cf. [15, Chapter 21]. However, Holmgren's theorem concerns only linear equations and this corresponds to an important difference between the holomorphic and the pseudo-holomorphic equations. In the former case, in local coordinates, the operator J is constant so the Cauchy–Riemann equation is indeed linear. Holmgren's theorem thus applies to give an alternative proof of the uniqueness of geodesics and geodesic rays. In general almost complex manifolds, instead, the Cauchy–Riemann equation is not locally linear.

3.5 Example: the 1-dimensional case, continued

We now take a closer look at the notion of geodesic convexity by exhibiting an example of a convex functional in the 1-dimensional case. This functional is actually very well known: it is the standard length functional. Its convexity is a rather striking fact, and it is well worth emphasizing it by giving two proofs. The first relies on the specific nature of the geodesic equation by bringing into play basic holomorphic function theory. As above, it assumes we have reparametrized the curve by integrating $f\partial\theta$, but it requires that the domain remains compact.

This first proof also leads to a monotonicity result for the length functional. The second proof is a direct geometric calculation, and holds for all f.

Proposition 3.9 *The length functional on closed curves in* $\mathbb{C}$ *is convex in the sense of Definition 2.3.*

Proof: For the first proof, assume we are given a holomorphic map on the cylinder

$$\gamma : \mathcal{S}^1_R \times (-\epsilon, \epsilon) \to \mathbb{C},$$

where $\mathcal{S}^1_R = \mathbb{R}/2\pi R\mathbb{Z}$. Let $w = s + it$ denote the complex variable on the cylinder and $\lambda = \lambda(t) : (-\epsilon, \epsilon) \to \mathbb{R}$ the length of the curve $\gamma(\cdot, t)$. Explicitly,

$$\lambda = \int_0^{2\pi R} \left| \frac{\partial \gamma}{\partial s} \right| ds = \int_0^{2\pi R} \left| \frac{\partial \gamma}{\partial w} \right| ds.$$

We want to prove that λ is convex with respect to t.

The biholomorphism $z = \phi(s,t)$ in (4) allows us to reformulate the problem in terms of a map $g = g(z) : A \to \mathbb{C}$ such that $\gamma = g \circ \phi$. Notice that g is holomorphic if and only if γ is holomorphic and their complex derivatives satisfy $|\frac{\partial \gamma}{\partial w}| = (1/R)|\phi \frac{\partial g}{\partial z}|$. Setting, using polar coordinates on $\mathbb{C}$,

$$\Lambda(r) := \int_0^{2\pi} \left| z \frac{\partial g}{\partial z} \right| d\theta,$$

it follows that $\lambda(t) = \Lambda(e^{-t/R})$. It thus suffices to prove that $\Lambda \circ \exp$ is convex.

Notice that the function $z\frac{\partial g}{\partial z}$ is holomorphic, so its norm $|z\frac{\partial g}{\partial z}| = r|\frac{\partial g}{\partial z}|$ is a subharmonic function on the annulus. Convexity then follows from a classical result due to Riesz: we refer to [6, Theorem 1.6] and the remark following that theorem for details. Actually, using that theorem we can further show that if g extends to a holomorphic function on the disk then $\Lambda(r)$ is non-decreasing. In terms of t, this shows that $\lambda(t)$ is non-increasing.

For the second proof, let us start by parametrizing the curve by arclength: it is thus the image of some map $\gamma_0(s)$, where $s \in L$. Choose a vector field $X = f\partial s$ and let $\gamma(s,t) = \gamma_t(s)$ be a family of curves satisfying the corresponding geodesic equation (1). Set $\gamma' := \frac{\partial \gamma}{\partial s}$ and $\dot{\gamma} := \frac{\partial \gamma}{\partial t}$ for a cleaner exposition.

The length functional along γ_t is given by

$$\lambda(\gamma_t) = \int_L |\gamma_t'| \mathrm{d}s.$$

We first calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(\gamma_t) = \int_L \frac{\partial}{\partial t} \langle \gamma_t', \gamma_t' \rangle^{\frac{1}{2}} \mathrm{d}s = \int_L |\gamma_t'|^{-1} \langle \dot{\gamma_t}', \gamma_t' \rangle \mathrm{d}s.$$

Then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda(\gamma_t) = \int_L |\gamma_t'|^{-1}\left(\langle \ddot{\gamma}_t{}', \gamma_t'\rangle + |\dot{\gamma}_t{}'|^2\right) - |\gamma_t'|^{-3}\langle \dot{\gamma}_t{}', \gamma_t'\rangle^2 \mathrm{d}s.$$

Since $|\gamma_t'|^{-1}\gamma_t'$ is a unit tangent vector we have that

$$\frac{\partial}{\partial s}\left(|\gamma_t'|^{-1}\gamma_t'\right) = |\gamma_t'|^{-1}\gamma_t'' - |\gamma_t'|^{-3}\langle\gamma_t'', \gamma_t'\rangle\gamma_t' = i\kappa_t|\gamma_t'|^{-1}\gamma_t',$$

where κ_t is the curvature of γ_t. Therefore,

$$\gamma_t'' = |\gamma_t'|^{-2}\langle\gamma_t'', \gamma_t'\rangle\gamma_t' + i\kappa_t\gamma_t'.$$

Hence,

$$\dot{\gamma}_t{}' = \frac{\partial}{\partial s}\dot{\gamma}_t = \frac{\partial}{\partial s}(if\gamma_t') = if\gamma_t'' + if'\gamma_t' = -f\kappa_t\gamma_t' + i(f' + f|\gamma_t'|^{-2}\langle\gamma_t'', \gamma_t'\rangle)\gamma_t'.$$

Moreover,

$$\ddot{\gamma}_t = \frac{\partial}{\partial t}(if\gamma_t') = if\dot{\gamma}_t{}' = -f(f' + f|\gamma_t'|^{-2}\langle\gamma_t'', \gamma_t'\rangle)\gamma_t' - if^2\kappa_t\gamma_t'.$$

Since we will be taking no further t derivatives and γ_0 was arbitrary we can now set $t = 0$ without loss of generality. In this case, because $|\gamma_0'| = 1$ we see that $\langle\gamma_0'', \gamma_0'\rangle = 0$ and thus $\gamma_0'' = i\kappa\gamma_0'$ where κ_0 is the curvature of γ_0. Hence,

$$\dot{\gamma}_t{}'|_{t=0} = -f\kappa_0\gamma_0' + if'\gamma_0'$$

and

$$\ddot{\gamma}_t|_{t=0} = -ff'\gamma_0' - if^2\kappa_0\gamma_0'.$$

We therefore see that

$$\begin{aligned}\ddot{\gamma}_t{}'|_{t=0} &= -(ff')'\gamma_0' - ff'\gamma_0'' - i(f^2\kappa_0)'\gamma_0' - if^2\kappa_0\gamma_0'' \\ &= (f^2\kappa_0^2 - (ff')')\gamma_0' - i(ff'\kappa_0 + (f^2\kappa_0)')\gamma_0'.\end{aligned}$$

Putting these formulae together we see that

$$\begin{aligned}\langle\ddot{\gamma}_t{}'', \gamma_t'\rangle|_{t=0} &= f^2\kappa_0^2 - (ff')' \\ |\dot{\gamma}_t{}'|^2|_{t=0} &= f^2\kappa_0^2 + (f')^2 \\ \langle\dot{\gamma}_t{}', \gamma_t'\rangle^2 &= f^2\kappa_0^2.\end{aligned}$$

We deduce that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda(\gamma_t)|_{t=0} &= \int_L f^2\kappa_0^2 - (ff')' + f^2\kappa_0^2 + (f')^2 - f^2\kappa_0^2\,\mathrm{d}s \\ &= \int_L (f')^2 + f^2\kappa_0^2\,\mathrm{d}s \geq 0\end{aligned}$$

since $\int_L (ff')'\mathrm{d}s = 0$. Therefore the length $\lambda(\gamma_t)$ is a convex function of t. $\square$

4 A canonical volume functional

In higher dimensions the standard, Riemannian, volume functional is not convex with respect to our notion of geodesics. This is hardly surprising: when $n \geq 2$ the totally real condition is an extra assumption on the submanifold, but the standard volume functional does not interact with this condition in any way. The goal of this section is to show that, for totally real submanifolds, there exists an alternative volume functional which (i) is canonical, (ii) does depend on the totally real condition and (iii) is convex in our sense in certain situations.

To define this functional we will need an alternative characterization of totally real planes in T_pM, as follows: an n-plane π in T_pM is totally real if and only if $\alpha|_\pi \neq 0$ for all (equivalently, for any) $\alpha \in K_M(p) \setminus \{0\}$, where K_M is the canonical bundle of (M, J).

Notice that n-planes π in T_pM which are not totally real contain a complex line: that is, a pair $\{X, JX\}$ for some $X \in T_pM \setminus \{0\}$. We call such n-planes *partially complex*. We then say that an n-dimensional submanifold is *partially complex* if this condition holds in the strongest sense possible, *i.e.* if each of its tangent spaces is partially complex.

Let TR^+ denote the Grassmannian bundle of oriented totally real n-planes in TM and let $\pi \in TR^+(p)$. Let $v_1, \dots, v_n$ be a positively oriented basis of π. We can then define $v_j^* \in T_p^*M \otimes \mathbb{C}$ by

$$v_j^*(v_k) = \delta_{jk} \quad \text{and} \quad v_j^*(Jv_k) = i\delta_{jk}.$$

This allows us to define a non-zero form $v_1^* \wedge \ldots \wedge v_n^* \in K_M(p)$.

As it stands, the form we have constructed depends on the choice of basis $v_1, \dots, v_n$. We can fix this by assuming that we have a Hermitian metric h on K_M, and then define

$$\sigma[\pi] = \frac{v_1^* \wedge \ldots \wedge v_n^*}{|v_1^* \wedge \ldots \wedge v_n^*|_h} \in K_M(p).$$

This form has unit norm and is now independent of the choice of basis.

We have thus defined a map between bundles $\sigma : TR^+ \to K_M$ covering the identity map on M (in fact, σ maps into the unit circle bundle in K_M). We also see that if we restrict the form $\sigma[\pi]$ to π we get a *real-valued* n-form.

Now let $\iota : L \to M$ be an n-dimensional totally real immersion. We can then obtain global versions of the above constructions as follows.

Canonical bundle over L. Let $K_M[\iota]$ denote the pullback of K_M over L, so the fibre of $K_M[\iota]$ over $p \in L$ is the fibre of K_M over $\iota(p) \in M$. This defines a complex line bundle over L which depends on ι.

We now observe that any complex-valued n-form α on T_pL defines a unique n-form $\widetilde{\alpha}$ on $T_{\iota(p)}M$ by identifying T_pL with its image via ι_* and setting, *e.g.*,

$$\widetilde{\alpha}[\iota(p)](J\iota_*(v_1), \dots, J\iota_*(v_n)) := i^n \alpha[p](v_1, \dots, v_n).$$

The totally real condition implies that this is an isomorphism: the bundle $K_M[\iota]$ is canonically isomorphic, via ι_*, with the (ι-independent) bundle $\Lambda^n(L,\mathbb{C}) := \Lambda^n(L,\mathbb{R})\otimes\mathbb{C}$ of complex-valued n-forms on L.

Canonical section. Now we use the fact that L is oriented. This implies that $\Lambda^n(L,\mathbb{R})$ is trivial, so $K_M[\iota]$ also is. We can build a global section of $K_M[\iota]$ using our previous linear-algebraic construction: $p\mapsto \Omega_J[\iota](p) := \sigma[\iota_*(T_pL)]$. We call $\Omega_J[\iota]$ the *canonical section* of $K_M[\iota]$. If we restrict the form $\Omega_J[\iota]$ to $\iota_*(T_pL)$ we obtain a *real-valued* positive n-form on $\iota(L)$, thus a volume form $\mathrm{vol}_J[\iota] := \iota^*(\Omega_J[\iota])$ on L: we call it the *J-volume form* of L, defined by ι.

When the totally real submanifold is compact we obtain a "canonical volume" $\int_L \mathrm{vol}_J[\iota]$, for $\iota\in\mathcal{P}$. One may see that if $\varphi\in\mathrm{Diff}(L)$ then $\mathrm{vol}_J[\iota\circ\varphi] = \varphi^*(\mathrm{vol}_J[\iota])$, just as for the standard volume form, thus

$$\int_L \mathrm{vol}_J[\iota\circ\varphi] = \int_L \varphi^*\,\mathrm{vol}_J[\iota] = \int_L \mathrm{vol}_J[\iota].$$

Hence the canonical volume descends to $\mathcal{T}$ to define the *J-volume functional*

$$\mathrm{Vol}_J : \mathcal{T}\to\mathbb{R},\quad L\mapsto \int_L \mathrm{vol}_J[\iota],$$

where ι is any parametrization representing the submanifold L.

Already in this context it would be possible to study its first variation, thus characterizing its critical points. Using the connection on $\mathcal{T}$ one could also define its second variation, studying the stability properties of the critical points. We will do this below, in the presence of additional structure and hypotheses on M which will allow us to determine a useful expression for the first variation and a simplified formula for the second variation.

Notation. From now on we will sometimes simplify notation by dropping the reference to the specific immersion used. Since this is standard practice in other contexts, *e.g.* when discussing the standard Riemannian volume, we expect it will not create any confusion.

5 The J-volume in the Hermitian context

Let us now assume that (M,J) is almost Hermitian, *i.e.* we choose a Riemannian metric $\overline{g}$ on M which is compatible with J, so J is an isometry defining a Hermitian metric h on M. Let us also choose a unitary connection $\widetilde{\nabla}$ on M.

Let L be an oriented totally real submanifold of (M,J). In Riemannian geometry it is customary to work in terms of the tangential and normal projections π_{T}, $\pi_\perp$ and with the Levi-Civita connection $\overline{\nabla}$. This choice however does not make any use of the totally real condition which implies that, for any $p\in L$, any vector $Z\in T_pM$ can alternatively be written uniquely as $Z = X + JY$ where $X,Y\in T_pL$. This splitting induces projections π_L,π_J by setting $\pi_L(Z) = X$ and $\pi_J(Z) = JY$: one should think of these as the natural projections in this context. The following fact is a simple computation.

Lemma 5.1 *$\pi_L \circ J = J \circ \pi_J$ and $J \circ \pi_L = \pi_J \circ J$.*

The structures on M induce structures h, $\widetilde{\nabla}$ on K_M, which we can use to define the J-volume form on L.

Notice that, in contrast to the previous section where we were given only a complex structure, we now have the 2-form $\overline{\omega}(\cdot,\cdot) = \overline{g}(J\cdot,\cdot)$. In this section we can thus also discuss Lagrangian submanifolds, defined by the condition $\iota^*\bar{\omega} = 0$.

5.1 The J-volume versus the Riemannian volume

In the almost Hermitian context, given an immersion ι, we can define the usual Riemannian volume form vol_g, using the induced metric g. It is useful to compare this with the J-volume form, cf. also [11].

Let $e_1, \ldots, e_n$ be a positive orthonormal basis for π and set $h_{ij} = h(e_i, e_j)$, where h is the ambient Hermitian metric. We wish to calculate $|e_1^* \wedge \ldots \wedge e_n^*|_h$. Observe that $h(., e_j) = h_{kj} e_k^*$ since

$$h(e_i, e_j) = h_{ij} = h_{kj} e_k^*(e_i)$$

and

$$h(Je_i, e_j) = ih_{ij} = ih_{kj} e_k^*(e_i) = h_{kj} e_k^*(Je_i).$$

Thus

$$h(., e_1) \wedge \ldots \wedge h(., e_n) = (\det_{\mathbb{C}} h_{ij}) e_1^* \wedge \ldots \wedge e_n^*.$$

Hence

$$|e_1^* \wedge \ldots \wedge e_n^*|_h = (\det_{\mathbb{C}} h_{ij})^{-1} |h(\cdot, e_1) \wedge \ldots \wedge h(\cdot, e_n)|_h.$$

We now notice that

$$|h(\cdot, e_1) \wedge \ldots \wedge h(\cdot, e_n)|_h^2 = \det_{\mathbb{C}} h_{ij}$$

so

$$|e_1^* \wedge \ldots \wedge e_n^*|_h = (\det_{\mathbb{C}} h_{ij})^{-\frac{1}{2}}. \tag{5}$$

We therefore find that

$$\mathrm{vol}_J = \frac{e_1^* \wedge \ldots \wedge e_n^*}{|e_1^* \wedge \ldots \wedge e_n^*|_h} |_\pi = (\det_{\mathbb{C}} h_{ij})^{1/2} \,\mathrm{vol}_g\,.$$

We can now obtain a well-defined function

$$\rho_J : TR^+ \to \mathbb{R}, \quad \rho_J(\pi) := \mathrm{vol}_J(e_1, \ldots, e_n) = (\det_{\mathbb{C}} h_{ij})^{1/2},$$

because this quantity is independent of the orthonormal basis $e_1, \ldots, e_n$.

Restricting this function to an oriented totally real submanifold L, we obtain the identity: $\mathrm{vol}_J = \rho_J \,\mathrm{vol}_g$.

Notice that $h = \overline{g} - i\overline{\omega}$ and that, using the obvious notation for the components of $\overline{g}$ and $\overline{\omega}$ with respect to $e_1, \ldots, e_n$,

$$\det{}_{\mathbb{C}} h_{ij} = \sqrt{\det \begin{pmatrix} \overline{g}_{ij} & \overline{\omega}_{ij} \\ -\overline{\omega}_{ij} & \overline{g}_{ij} \end{pmatrix}}.$$

We see that

$$\overline{\omega}_{ij} = \overline{g}(Je_i, e_j) \quad \text{and} \quad -\overline{\omega}_{ij} = \overline{g}(e_i, Je_j).$$

We deduce that $\det_{\mathbb{C}} h_{ij} = \sqrt{\det(\overline{g}_{ab})}$ where $\overline{g}_{ab}$ is the matrix of $\overline{g}$ with respect to the basis $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$. Therefore

$$\det{}_{\mathbb{C}} h_{ij} = \mathrm{vol}_{\overline{g}}(e_1, \ldots, e_n, Je_1, \ldots, Je_n).$$

We deduce a second expression for ρ_J:

$$\rho_J(\pi) = \sqrt{\mathrm{vol}_{\overline{g}}(e_1, \ldots, e_n, Je_1, \ldots, Je_n)}.$$

Hence we see that $\rho_J(\pi) \leq 1$ with equality if and only if π is Lagrangian.

More generally, given any basis $v_1, \ldots, v_n$ for π, we can write

$$\rho_J(\pi) = \frac{\sqrt{\mathrm{vol}_{\overline{g}}(v_1, \ldots, v_n, Jv_1, \ldots, Jv_n)}}{|v_1 \wedge \cdots \wedge v_n|_{\overline{g}}}.$$

We can set $\rho_J(\pi) = 0$ when π is partially complex and extend the map σ to all n-planes, just setting $\sigma[\pi] = 0$ if π is partially complex. This is particularly reasonable in this almost Hermitian setting, where there is a natural topology on the Grassmannian of n-planes: this choice of extension of σ would be justified by the fact that it is the unique one which preserves the continuity of σ.

Applying these observation to submanifolds we deduce the following.

Lemma 5.2 *For any compact oriented n-dimensional submanifold L in an almost Hermitian manifold $(M, J, \overline{g})$, we have $\mathrm{Vol}_J(L) \leq \mathrm{Vol}_g(L)$ with equality if and only if L is Lagrangian. In particular, if a Lagrangian minimizes the J-volume functional then it minimizes the volume functional.*

Proof: If L is a compact Lagrangian minimizing the J-volume functional and L' is any other compact oriented n-dimensional submanifold then

$$\mathrm{Vol}_g(L') \geq \mathrm{Vol}_J(L') \geq \mathrm{Vol}_J(L) = \mathrm{Vol}_g(L).$$

The result follows. $\square$

5.2 First variation of the J-volume

Proposition 5.3 *Let $\iota_t : L \to L_t \subseteq M$ be a one-parameter family of totally real submanifolds in an almost Hermitian manifold and let $\frac{\partial}{\partial t}\iota_t|_{t=0} = Z$.*

Set $\iota = \iota_0$ and $g = \iota^*\overline{g}$. If $Z = X + JY$ for tangent vectors X, Y then

$$\begin{aligned}\frac{\partial}{\partial t}\operatorname{vol}_J[\iota_t]|_{t=0} &= \overline{g}\big(\pi_L\big(\widetilde{\nabla}_{e_i} Z + \widetilde{T}(Z, e_i)\big), e_i\big)\operatorname{vol}_J[\iota] \\ &= \operatorname{div}(\rho_J X)\operatorname{vol}_g + \overline{g}\big(\pi_L\big(\widetilde{\nabla}_{e_i} JY + \widetilde{T}(JY, e_i)\big), e_i\big)\operatorname{vol}_J[\iota],\end{aligned}$$

where at $p \in L$ we have that $e_1, \ldots, e_n$ is a g-orthonormal basis for T_pL.

Notice that the quantities which appear in the above formulae are invariant under changes of orthonormal basis and so are globally defined.

Proof: Let $p \in L$, let $x_1, \ldots, x_n$ be normal coordinates at p with respect to g and let $e_i = \frac{\partial}{\partial x_i}$: this defines an orthonormal basis for T_pL. Since we have $L_t = \iota_t(L) = \iota(L, t)$ where $\iota(x, t) := \iota_t(x)$, we may consider coordinates $(x_1, \ldots, x_n, t)$ near $(p, 0)$ in $L \times (-\epsilon, \epsilon)$, so then $\iota_*(\frac{\partial}{\partial t}) = Z$ at $t = 0$. Notice that since $(x_1, \ldots, x_n, t)$ is a coordinate system on $L \times (-\epsilon, \epsilon)$ we see that $[Z, e_i] = 0$.

Recall that the J-volume form is given at p by

$$\operatorname{vol}_J[\iota] = \rho_J(\pi)\operatorname{vol}_g = \sqrt{\operatorname{vol}_{\overline{g}}(e_1, \ldots, e_n, Je_1, \ldots, Je_n)}\operatorname{vol}_g,$$

where $\pi = T_{\iota(p)}L$. We wish to calculate

$$\frac{\partial}{\partial t}\operatorname{vol}_J[\iota_t]|_{t=0} = \frac{\partial}{\partial t}\sqrt{\operatorname{vol}_{\overline{g}}(e_1(t), \ldots, e_n(t), Je_1(t), \ldots, Je_n(t))}\operatorname{vol}_{g_t}|_{t=0}$$

at p, where $e_1(t), \ldots, e_n(t)$ is orthonormal at $\iota_t(p) \in L_t$ with respect to $g_t = \iota_t^*\overline{g}$ and $e_i(0) = e_i$ for all i. This computation can be simplified by noticing the alternative expression

$$\operatorname{vol}_J[\iota_t] = \rho_J(t)\operatorname{vol}_g,$$

where $\rho_J(t) = \sqrt{\operatorname{vol}_{\overline{g}}((\iota_t)_*e_1, \ldots, (\iota_t)_*e_n, J(\iota_t)_*e_1, \ldots, J(\iota_t)_*e_n)}$. Indeed, setting $\pi_t = T_{\iota_t(p)}L_t$ and using our previous expressions for ρ_J, we find

$$\begin{aligned}&\operatorname{vol}_J[\iota_t](e_1, \ldots, e_n) \\ &\quad = \rho_J(\pi_t)\operatorname{vol}_{g_t}(e_1, \ldots, e_n) \\ &\quad = \frac{\sqrt{\operatorname{vol}_{\overline{g}}((\iota_t)_*e_1, \ldots, (\iota_t)_*e_n, J(\iota_t)_*e_1, \ldots, J(\iota_t)_*e_n)}}{|(\iota_t)_*e_1 \wedge \cdots \wedge (\iota_t)_*e_n|_{\overline{g}}} \cdot |e_1 \wedge \cdots \wedge e_n|_{\iota_t^*\overline{g}} \\ &\quad = \sqrt{\operatorname{vol}_{\overline{g}}((\iota_t)_*e_1, \ldots, (\iota_t)_*e_n, J(\iota_t)_*e_1, \ldots, J(\iota_t)_*e_n)}.\end{aligned}$$

It therefore suffices to compute

$$
\begin{aligned}
&\frac{\partial}{\partial t}\rho_J(t)\\
&= \frac{\sum_{i=1}^n \mathrm{vol}_{\overline{g}}((\iota_t)_* e_1, \dots, \frac{\partial}{\partial t}(\iota_t)_* e_i, \dots, (\iota_t)_* e_n, J(\iota_t)_* e_1, \dots, J(\iota_t)_* e_n)}{2\rho_J(t)}\\
&+ \frac{\sum_{i=1}^n \mathrm{vol}_{\overline{g}}((\iota_t)_* e_1, \dots, (\iota_t)_* e_n, J(\iota_t)_* e_1, \dots, \frac{\partial}{\partial t} J(\iota_t)_* e_i, \dots, J(\iota_t)_* e_n)}{2\rho_J(t)}. \qquad (6)
\end{aligned}
$$

Setting $t = 0$ so that $\frac{\partial}{\partial t}|_{t=0} = \widetilde{\nabla}_Z$ and the denominator becomes $2\rho_J$ we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho_J(t)|_{t=0} &= \frac{\sum_{i=1}^n \mathrm{vol}_{\overline{g}}(e_1, \dots, \widetilde{\nabla}_Z\, e_i, \dots, e_n, Je_1, \dots, Je_n)}{2\rho_J}\\
&+ \frac{\sum_{i=1}^n \mathrm{vol}_{\overline{g}}(e_1, \dots, e_n, Je_1, \dots, \widetilde{\nabla}_Z\, Je_i, \dots, Je_n)}{2\rho_J}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\sum_{i=1}^n \mathrm{vol}_{\overline{g}}(e_1, \dots, \widetilde{\nabla}_Z\, e_i, \dots, e_n, Je_1, \dots, Je_n)\\
&\qquad = \sum_{i=1}^n \mathrm{vol}_{\overline{g}}(e_1, \dots, \pi_L\, \widetilde{\nabla}_Z\, e_i, \dots, e_n, Je_1, \dots, Je_n)\\
&\qquad = \sum_{i=1}^n \overline{g}(\pi_L\, \widetilde{\nabla}_Z\, e_i, e_i)\, \mathrm{vol}_{\overline{g}}(e_1, \dots, e_n, Je_1, \dots, Je_n)\\
&\qquad = \sum_{i=1}^n \overline{g}(\pi_L\, \widetilde{\nabla}_Z\, e_i, e_i)\rho_J^2.
\end{aligned}
$$

Similarly, using Lemma 5.1

$$
\begin{aligned}
&\sum_{i=1}^n \mathrm{vol}_{\overline{g}}(e_1, \dots, e_n, Je_1, \dots, \widetilde{\nabla}_Z\, Je_i, \dots, Je_n)\\
&\qquad = \sum_{i=1}^n \mathrm{vol}_{\overline{g}}(e_1, \dots, e_n, Je_1, \dots, \pi_J J\, \widetilde{\nabla}_Z\, e_i, \dots, Je_n)\\
&\qquad = \sum_{i=1}^n \overline{g}(\pi_J J\, \widetilde{\nabla}_Z\, e_i, Je_i)\, \mathrm{vol}_{\overline{g}}(e_1, \dots, e_i, \dots, e_n, Je_1, \dots, Je_n)\\
&\qquad = \sum_{i=1}^n \overline{g}(\pi_L\, \widetilde{\nabla}_Z\, e_i, e_i)\rho_J^2.
\end{aligned}
$$

Thus,

$$
\frac{\partial}{\partial t}\rho_J(t)|_{t=0} = \overline{g}(\pi_L\, \widetilde{\nabla}_Z\, e_i, e_i)\rho_J.
$$

Since $[Z,e_i]=0$, we have that

$$\widetilde{\nabla}_Z\, e_i = \widetilde{\nabla}_{e_i} Z + \tilde{T}(Z,e_i) + [Z,e_i] = \widetilde{\nabla}_{e_i} Z + \widetilde{T}(Z,e_i),$$

where $\widetilde{T}$ is the torsion of $\widetilde{\nabla}$. The first part of the result follows.

We see that for $Z = X$ tangential

$$\frac{\partial}{\partial t}\operatorname{vol}_J[\iota_t]|_{t=0} = \mathcal{L}_X \operatorname{vol}_J[\iota] = \mathrm{d}(\rho_J X \lrcorner \operatorname{vol}_g),$$

using Cartan's formula, which gives the result. $\square$

Now suppose that L is compact (without boundary). We can then define the J-volume of L as before by:

$$\operatorname{Vol}_J(L) = \int_L \operatorname{vol}_J = \int_L \rho_J \operatorname{vol}_g = \int_L \sqrt{\operatorname{vol}_{\overline{g}}(e_1,\ldots,e_n,Je_1,\ldots,Je_n)}\operatorname{vol}_g,$$

where $e_1,\ldots,e_n$ is an orthonormal basis for each tangent space.

We want to compute the first variation of the J-volume functional. By Proposition 5.3 if $Z = X + JY$ for X, Y tangential then

$$\frac{\partial}{\partial t}\operatorname{Vol}_J(L_t)|_{t=0} = \int_L \overline{g}\big(\pi_L\big(\widetilde{\nabla}_{e_i} JY + \widetilde{T}(JY,e_i)\big), e_i\big)\operatorname{vol}_J$$

since

$$\int_L \operatorname{div}(\rho_J X)\operatorname{vol}_g = 0$$

by Stokes' Theorem. Hence it is enough to restrict to $Z = JY$. Our final result will be phrased in terms of a "J-mean curvature vector field" defined as follows.

Let us use the metric $\bar{g}$ to define the transposed operators

$$\pi_J^t : T_pM \to (T_pL)^{\perp}, \quad \pi_L^t : T_pM \to (J(T_pL))^{\perp}.$$

Observe that $(J(T_pL))^{\perp} = J(T_pL)^{\perp}$ since $X \in (J(T_pL))^{\perp}$ if and only if for all $Y \in T_pL$,

$$\overline{g}(Y, JX) = -\overline{g}(JY, X) = 0,$$

which means $JX \in (T_pL)^{\perp}$ and thus $X \in J(T_pL)^{\perp}$. Then, using the tangential projection π_T defined using $\overline{g}$, one may check that

$$\pi_T J\,\widetilde{\nabla}\,\pi_L^t : T_pL \times T_pL \to T_pL$$

is C^∞-bilinear on its domain, so it is a tensor and its trace is a well-defined vector on L. We now set

$$H_J := -J(\operatorname{tr}_L(\pi_T J\,\widetilde{\nabla}\,\pi_L^t)). \tag{7}$$

This is a well-defined vector field on L, taking values in the bundle $J(TL)$. We refer to [11] for an alternative expression for H_J.

Proposition 5.4 *Let $\iota_t : L \to L_t \subseteq M$ be compact totally real submanifolds in an almost Hermitian manifold and let $\frac{\partial}{\partial t}\iota_t|_{t=0} = X + JY$ for X, Y tangential. Then*

$$\frac{\partial}{\partial t}\operatorname{Vol}_J(L_t)|_{t=0} = -\int_L \overline{g}(JY, H_J + S_J)\operatorname{vol}_J$$

where, given $p \in L$ and an orthonormal basis $e_1, \ldots, e_n$ for T_pL we have H_J given by (7) *and*

$$S_J = -\overline{g}(\pi_L\widetilde{T}(Je_j, e_i), e_i)Je_j. \tag{8}$$

Proof: We calculate that

$$\begin{aligned}
&\overline{g}\big(\pi_L\big(\widetilde{\nabla}_{e_i} JY + \widetilde{T}(JY, e_i)\big), e_i\big)\\
&\qquad = \overline{g}(\widetilde{\nabla}_{e_i} JY + \widetilde{T}(JY, e_i), \pi_L^{\mathrm{t}} e_i)\\
&\qquad = -\overline{g}(JY, \widetilde{\nabla}_{e_i} \pi_L^{\mathrm{t}} e_i) + \overline{g}(\widetilde{T}(\overline{g}(JY, Je_j)Je_j, e_i), \pi_L^{\mathrm{t}} e_i)\\
&\qquad = -\overline{g}(JY, \widetilde{\nabla}_{e_i} \pi_L^{\mathrm{t}} e_i) + \overline{g}(JY, Je_j)\overline{g}(\widetilde{T}(Je_j, e_i), \pi_L^{\mathrm{t}} e_i)\\
&\qquad = -\overline{g}(JY, \widetilde{\nabla}_{e_i} \pi_L^{\mathrm{t}} e_i) + \overline{g}(JY, \overline{g}\big(\widetilde{T}(Je_j, e_i), \pi_L^{\mathrm{t}} e_i)Je_j)\\
&\qquad = -\overline{g}\Big(JY, -J\pi_{\mathrm{T}}J\big(\widetilde{\nabla}_{e_i} \pi_L^{\mathrm{t}} e_i - \overline{g}\big(\pi_L\widetilde{T}(Je_j, e_i), e_i)Je_j\big)\Big)\\
&\qquad = -\overline{g}\Big(JY, H_J - \overline{g}\big(\pi_L\widetilde{T}(Je_j, e_i), e_i)Je_j\Big)
\end{aligned}$$

by definition of H_J and the fact that $-J\pi_T J(JX) = JX$. The formula follows since the term involving X integrates to 0. $\square$

Remark If we define a Riemannian metric G on $\mathcal{T}$ by

$$G_L(JX, JY) = \int_L \overline{g}(X, Y)\operatorname{vol}_J$$

for $JX, JY \in T_L\mathcal{T}$ then the downward gradient vector field of the J-volume functional Vol_J on $\mathcal{T}$ with respect to G is given by $H_J + S_J$. The corresponding flow (the "J-mean curvature flow") is studied in [11].

5.3 Second variation of the J-volume

We now wish to study the important question of stability of critical points of the J-volume, so we calculate the second variation of the J-volume form. This generalises calculations in [1, Proposition 3], which built on an original result for the second variation of volume of Lagrangians in Kähler manifolds by Chen, Leung and Nagano [2, Theorem 4.1].

Proposition 5.5 *Let $\iota_{s,t} : L \to L_{s,t} \subseteq M$ be a two-parameter family of totally real submanifolds in an almost Hermitian manifold and let $\frac{\partial}{\partial s}\iota_{s,t}|_{s=t=0} = W$*

and $\frac{\partial}{\partial t}\iota_{s,t}|_{s=t=0} = Z$. *Then*

$$
\begin{aligned}
&\frac{\partial^2}{\partial s\partial t}\operatorname{vol}_J[\iota_{s,t}]|_{s=t=0}\\
&= \Big(\overline{g}(\pi_L J(\widetilde{\nabla}_{e_i} W + \widetilde{T}(W,e_i)), e_j)\overline{g}(\pi_L J(\widetilde{\nabla}_{e_j} Z + \widetilde{T}(Z,e_j)), e_i)\\
&- \overline{g}(\pi_L(\widetilde{\nabla}_{e_i} W + \widetilde{T}(W,e_i)), e_j)\overline{g}(\pi_L(\widetilde{\nabla}_{e_j} Z + \widetilde{T}(Z,e_j)), e_i)\\
&+ \overline{g}\big(\pi_L\big(\widetilde{\nabla}_{e_i} W + \widetilde{T}(W,e_i)\big), e_i\big)\overline{g}\big(\pi_L\big(\widetilde{\nabla}_{e_j} Z + \widetilde{T}(Z,e_j)\big), e_j\big)\\
&+ \overline{g}(\pi_L(\widetilde{R}(W,e_i)Z + \widetilde{\nabla}_{e_i}\widetilde{\nabla}_W Z), e_i)\\
&+ \overline{g}(\pi_L(\widetilde{\nabla}_W \widetilde{T}(Z,e_i) + \widetilde{T}(\widetilde{\nabla}_W Z, e_i) + \widetilde{T}(Z, \widetilde{\nabla}_{e_i} W + \widetilde{T}(W,e_i))), e_i)\Big)\operatorname{vol}_J[\iota],
\end{aligned}
$$

where at $p \in L$ *we have that* $e_1, \ldots, e_n$ *is an orthonormal basis for* T_pL.

Notice again that these quantities are independent of the orthonormal basis chosen for T_pL and hence are globally defined.

Proof: Let $p \in L$. We choose coordinates $(x_1, \ldots, x_n, s, t)$ on $L \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ in a similar manner to the proof of Proposition 5.3, so that $(x_1, \ldots, x_n)$ are normal coordinates at p, $e_i = \frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial t}$ pushforward to W and Z respectively at $t = 0$. Hence $[W, e_i] = [Z, e_i] = 0$, as before.

Moreover, as in the proof of Proposition 5.3 we see that

$$\operatorname{vol}_J[\iota_{s,t}] = \rho_J(s,t)\operatorname{vol}_g$$

where

$$\rho_J(s,t) = \sqrt{\operatorname{vol}_{\overline{g}}((\iota_{s,t})_* e_1, \ldots, (\iota_{s,t})_* e_n, J(\iota_{s,t})_* e_1, \ldots, J(\iota_{s,t})_* e_n)}.$$

By (6) we have that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\rho_J(s,t)^2\\
&= \sum_{i=1}^n \operatorname{vol}_{\overline{g}}((\iota_{s,t})_* e_1, \ldots, \frac{\partial}{\partial t}(\iota_{s,t})_* e_i, \ldots, (\iota_{s,t})_* e_n, J(\iota_{s,t})_* e_1, \ldots, J(\iota_{s,t})_* e_n)\\
&+ \sum_{i=1}^n \operatorname{vol}_{\overline{g}}((\iota_{s,t})_* e_1, \ldots, (\iota_{s,t})_* e_n, J(\iota_{s,t})_* e_1, \ldots, \frac{\partial}{\partial t}J(\iota_{s,t})_* e_i, \ldots, J(\iota_{s,t})_* e_n).
\end{aligned}
$$

Hence we may calculate, substituting $\widetilde{\nabla}_W$ for $\frac{\partial}{\partial s}|_{s=t=0}$ and $\widetilde{\nabla}_Z$ for $\frac{\partial}{\partial t}|_{s=t=0}$:

$$
\begin{aligned}
&\frac{\partial^2}{\partial s \partial t}\rho_J(s,t)^2|_{s=t=0} \\
&\qquad = \sum_{i\neq j} \mathrm{vol}_{\overline{g}}(e_1,\ldots,\widetilde{\nabla}_Z\, e_i,\ldots,\widetilde{\nabla}_W\, e_j,\ldots,e_n,Je_1,\ldots,Je_n) \\
&\qquad + \sum_{i=1}^{n} \mathrm{vol}_{\overline{g}}(e_1,\ldots,\widetilde{\nabla}_W\,\widetilde{\nabla}_Z\, e_i,\ldots,e_n,Je_1,\ldots,Je_n) \\
&\qquad + \sum_{i,j=1}^{n} \mathrm{vol}_{\overline{g}}(e_1,\ldots,\widetilde{\nabla}_Z\, e_i,\ldots,e_n,Je_1,\ldots,\widetilde{\nabla}_W\, Je_j,\ldots,Je_n) \\
&\qquad + \sum_{i\neq j} \mathrm{vol}_{\overline{g}}(e_1,\ldots,e_n,Je_1,\ldots,\widetilde{\nabla}_Z\, Je_i,\ldots,\widetilde{\nabla}_W\, Je_j,\ldots,Je_n) \\
&\qquad + \sum_{i=1}^{n} \mathrm{vol}_{\overline{g}}(e_1,\ldots,e_n,Je_1,\ldots,\widetilde{\nabla}_W\,\widetilde{\nabla}_Z\, Je_i,\ldots,Je_n) \\
&\qquad + \sum_{i,j=1}^{n} \mathrm{vol}_{\overline{g}}(e_1,\ldots,\widetilde{\nabla}_W\, e_i,\ldots,e_n,Je_1,\ldots,\widetilde{\nabla}_Z\, Je_j,\ldots,Je_n).
\end{aligned}
$$

The first and fourth terms both give

$$
\begin{aligned}
&\sum_{i\neq j}\big(\overline{g}(\pi_L\,\widetilde{\nabla}_Z\, e_i,e_i)\overline{g}(\pi_L\,\widetilde{\nabla}_W\, e_j,e_j) - \overline{g}(\pi_L\,\widetilde{\nabla}_Z\, e_i,e_j)\overline{g}(\pi_L\,\widetilde{\nabla}_W\, e_j,e_i)\big)\rho_J^2 \\
&= \sum_{i,j}\big(\overline{g}(\pi_L\,\widetilde{\nabla}_Z\, e_i,e_i)\overline{g}(\pi_L\,\widetilde{\nabla}_W\, e_j,e_j) - \overline{g}(\pi_L\,\widetilde{\nabla}_Z\, e_i,e_j)\overline{g}(\pi_L\,\widetilde{\nabla}_W\, e_j,e_i)\big)\rho_J^2.
\end{aligned}
$$

The second and fifth terms both give

$$
\sum_{i=1}^{n}\overline{g}(\pi_L\,\widetilde{\nabla}_W\,\widetilde{\nabla}_Z\, e_i,e_i)\rho_J^2.
$$

Finally, the third and sixth terms both give

$$
\begin{aligned}
&\sum_{i,j}\big(\overline{g}(\pi_L\,\widetilde{\nabla}_Z\, e_i,e_i)\overline{g}(\pi_L\,\widetilde{\nabla}_W\, e_j,e_j) - \overline{g}(\pi_J\,\widetilde{\nabla}_Z\, e_i,Je_j)\overline{g}(\pi_L J\,\widetilde{\nabla}_W\, e_j,e_i)\big)\rho_J^2 \\
&= \sum_{i,j}\big(\overline{g}(\pi_L\,\widetilde{\nabla}_Z\, e_i,e_i)\overline{g}(\pi_L\,\widetilde{\nabla}_W\, e_j,e_j) + \overline{g}(\pi_L J\,\widetilde{\nabla}_Z\, e_i,e_j)\overline{g}(\pi_L J\,\widetilde{\nabla}_W\, e_j,e_i)\big)\rho_J^2.
\end{aligned}
$$

Clearly,

$$
\frac{\partial^2\rho_J(s,t)}{\partial s\partial t} = \frac{1}{2\rho_J(s,t)}\left(\frac{\partial^2\rho_J(s,t)^2}{\partial s\partial t} - 2\frac{\partial\rho_J(s,t)}{\partial s}\frac{\partial\rho_J(s,t)}{\partial t}\right)
$$

and

$$
\frac{\partial\rho_J(s,t)}{\partial s}\frac{\partial\rho_J(s,t)}{\partial t}|_{s=t=0} = \sum_{i,j}\overline{g}(\pi_L\,\widetilde{\nabla}_Z\, e_i,e_i)\overline{g}(\pi_L\,\widetilde{\nabla}_W\, e_j,e_j)\rho_J^2
$$

by (6). Hence,

$$\begin{aligned}
&\frac{\partial^2 \rho_J(s,t)}{\partial s \partial t}|_{s=t=0} \\
&\quad = \Big(\sum_{i,j} \overline{g}(\pi_L \widetilde{\nabla}_Z e_i, e_i) \overline{g}(\pi_L \widetilde{\nabla}_W e_j, e_j) - \sum_{i,j} \overline{g}(\pi_L \widetilde{\nabla}_Z e_i, e_j) \overline{g}(\pi_L \widetilde{\nabla}_W e_j, e_i) \\
&\qquad + \sum_{i,j} \overline{g}(\pi_L J \widetilde{\nabla}_Z e_i, e_j) \overline{g}(\pi_L J \widetilde{\nabla}_W e_j, e_i) + \sum_{i=1}^{n} \overline{g}(\pi_L \widetilde{\nabla}_W \widetilde{\nabla}_Z e_i, e_i) \Big) \rho_J.
\end{aligned}$$

Replacing $\widetilde{\nabla}_W e_i = \widetilde{\nabla}_{e_i} W + \widetilde{T}(W, e_i)$ (since $[W, e_i] = 0$) and similarly for Z gives the first three terms. For the last term, we see that

$$\begin{aligned}
\widetilde{\nabla}_W \widetilde{\nabla}_Z e_i &= \widetilde{\nabla}_W (\widetilde{\nabla}_{e_i} Z + \widetilde{T}(Z, e_i)) \\
&= \widetilde{R}(W, e_i) Z + \widetilde{\nabla}_{e_i} \widetilde{\nabla}_W Z \\
&\quad + (\widetilde{\nabla}_W \widetilde{T})(Z, e_i) + \widetilde{T}(\widetilde{\nabla}_W Z, e_i) + \widetilde{T}(Z, \widetilde{\nabla}_W e_i).
\end{aligned}$$

Again replacing $\widetilde{\nabla}_W e_i = \widetilde{\nabla}_{e_i} W + \widetilde{T}(W, e_i)$ gives the final term. □

A simple special case is as follows.

Corollary 5.6 *Use the notation of Proposition 5.5. If $W = Z = X$ tangential then*

$$\frac{\partial^2}{\partial t^2} \operatorname{vol}_J[\iota_t]|_{t=0} = \operatorname{div}(X \operatorname{div}(\rho_J X)) \operatorname{vol}_g .$$

Proof: We can see this from

$$\frac{\partial^2}{\partial t^2} \operatorname{vol}_J[\iota_t]|_{t=0} = \mathcal{L}_X \mathcal{L}_X \operatorname{vol}_J[\iota] = \mathrm{d}(X \lrcorner \mathrm{d}(\rho_J X \lrcorner \operatorname{vol}_g))$$

using Cartan's formula twice. □

A more important case is the following.

Proposition 5.7 *Use the notation of Propositions 5.4 and 5.5. If $W = Z = JY$ where Y is tangential and M is Kähler, then*

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} \operatorname{vol}_J[\iota_t]|_{t=0} &= - \operatorname{div}(Y \operatorname{div}(\rho_J Y)) \operatorname{vol}_g + \left(\frac{\operatorname{div}(\rho_J Y)}{\rho_J} \right)^2 \operatorname{vol}_J[\iota] \\
&\quad + \overline{g}(JY, H_J)^2 \operatorname{vol}_J[\iota] - \overline{\operatorname{Ric}}(Y, Y) \operatorname{vol}_J[\iota] \\
&\quad - \overline{g}(\pi_J(\overline{\nabla}_{JY} JY + \overline{\nabla}_Y Y), H_J) \operatorname{vol}_J[\iota] \\
&\quad + \operatorname{div} \big(\rho_J \pi_L (\overline{\nabla}_{JY} JY + \overline{\nabla}_Y Y) \big) \operatorname{vol}_g .
\end{aligned}$$

Proof: In this setting $\widetilde{\nabla} = \overline{\nabla}$, the Levi-Civita connection, and $\widetilde{T} = 0$. Hence by Proposition 5.5 we have

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\operatorname{vol}_J[\iota_t]|_{t=0} \\
&= \Big(\overline{g}(\pi_L J\overline{\nabla}_{e_i} JY, e_j)\overline{g}(\pi_L J\overline{\nabla}_{e_j} JY, e_i) - \overline{g}(\pi_L\overline{\nabla}_{e_i} JY, e_j)\overline{g}(\pi_L\overline{\nabla}_{e_j} JY, e_i) \\
&\qquad + \overline{g}\big(\pi_L\overline{\nabla}_{e_i} JY, e_i\big)\overline{g}\big(\pi_L\overline{\nabla}_{e_j} JY, e_j\big) \\
&\qquad + \overline{g}(\pi_L(\overline{R}(JY, e_i)JY + \overline{\nabla}_{e_i}\overline{\nabla}_{JY} JY), e_i)\Big)\operatorname{vol}_J[\iota].
\end{aligned} \tag{9}$$

Moreover, Proposition 5.5 and Corollary 5.6 give us that

$$\begin{aligned}
&\operatorname{div}(Y\operatorname{div}(\rho_J Y))\operatorname{vol}_g \\
&= \Big(\overline{g}(\pi_L J\overline{\nabla}_{e_i} Y, e_j)\overline{g}(\pi_L J\overline{\nabla}_{e_j} Y, e_i) - \overline{g}(\pi_L\overline{\nabla}_{e_i} Y, e_j)\overline{g}(\pi_L\overline{\nabla}_{e_j} Y, e_i) \\
&\qquad + \overline{g}\big(\pi_L\overline{\nabla}_{e_i} Y, e_i\big)\overline{g}\big(\pi_L\overline{\nabla}_{e_j} Y, e_j\big) + \overline{g}(\pi_L(\overline{R}(Y, e_i)Y + \overline{\nabla}_{e_i}\overline{\nabla}_Y Y), e_i)\Big)\operatorname{vol}_J[\iota].
\end{aligned}$$

We observe that, since $\overline{\nabla} J = 0$ as M is Kähler,

$$\begin{aligned}
&\overline{g}(\pi_L J\overline{\nabla}_{e_i}(JY), e_j)\overline{g}(\pi_L J\overline{\nabla}_{e_j}(JY), e_i) - \overline{g}(\pi_L\overline{\nabla}_{e_i}(JY), e_j)\overline{g}(\pi_L\overline{\nabla}_{e_j}(JY), e_i) \\
&= \overline{g}(\pi_L\overline{\nabla}_{e_i} Y, e_j)\overline{g}(\pi_L\overline{\nabla}_{e_j} Y, e_i) - \overline{g}(\pi_L J\overline{\nabla}_{e_i} Y, e_j)\overline{g}(\pi_L J\overline{\nabla}_{e_j} Y, e_i).
\end{aligned}$$

Hence, we see that

$$\begin{aligned}
\big(\overline{g}(\pi_L J\overline{\nabla}_{e_i}(JY), e_j)&\overline{g}(\pi_L J\overline{\nabla}_{e_j}(JY), e_i) \\
&- \overline{g}(\pi_L\overline{\nabla}_{e_i}(JY), e_j)\overline{g}(\pi_L\overline{\nabla}_{e_j}(JY), e_i)\big)\rho_J \\
&= -\operatorname{div}(Y\operatorname{div}\rho_J Y) + \overline{g}\big(\pi_L\overline{\nabla}_{e_i} Y, e_i\big)\overline{g}\big(\pi_L\overline{\nabla}_{e_j} Y, e_j\big)\rho_J \\
&\quad + \overline{g}(\pi_L(\overline{R}(Y, e_i)Y + \overline{\nabla}_{e_i}\overline{\nabla}_Y Y), e_i)\rho_J.
\end{aligned} \tag{10}$$

We see that, using the Kähler condition,

$$\begin{aligned}
\overline{g}(\pi_L\overline{R}(JY, e_i)JY, e_i) &= -\overline{g}(\pi_J\overline{R}(JY, e_i)Y, Je_i) \\
&= -\overline{g}(\overline{R}(Y, \pi_J^{\mathrm{t}} Je_i)JY, e_i) \\
&= \overline{g}(\overline{R}(Y, \pi_J^{\mathrm{t}} Je_i)Y, Je_i) \\
&= \overline{g}(\pi_J\overline{R}(Y, Je_i)Y, Je_i).
\end{aligned}$$

We deduce that

$$\begin{aligned}
\overline{g}(\pi_L\overline{R}(JY, e_i)JY, e_i) &+ \overline{g}(\pi_L\overline{R}(Y, e_i)Y, e_i) \\
&= \overline{g}(\pi_L\overline{R}(Y, e_i)Y, e_i) + \overline{g}(\pi_J\overline{R}(Y, Je_i)Y, Je_i) \\
&= -\overline{\operatorname{Ric}}(Y, Y).
\end{aligned}$$

Therefore, using (9) and (10) we see that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\operatorname{vol}_J[\iota_t]|_{t=0} = &-\operatorname{div}(Y\operatorname{div}(\rho_J Y))\operatorname{vol}_g + \overline{g}(\pi_L\overline{\nabla}_{e_i}Y, e_i)^2\operatorname{vol}_J[\iota]\\ &+\overline{g}(\pi_L\overline{\nabla}_{e_i}JY, e_i)^2\operatorname{vol}_J[\iota] - \overline{\operatorname{Ric}}(Y,Y)\operatorname{vol}_J[\iota]\\ &+\overline{g}(\pi_L\overline{\nabla}_{e_i}(\overline{\nabla}_{JY}JY+\overline{\nabla}_Y Y), e_i)\operatorname{vol}_J[\iota]\end{aligned}$$

As in Propositions 5.3 and 5.4, since here $\widetilde{T}=0$, we see that

$$\overline{g}(\pi_L\overline{\nabla}_{e_i}Y, e_i) = \frac{\operatorname{div}(\rho_J Y)}{\rho_J}, \qquad \overline{g}(\pi_L\overline{\nabla}_{e_i}JY, e_i) = -\overline{g}(JY, H_J),$$

and

$$\begin{aligned}\overline{g}(\pi_L\overline{\nabla}_{e_i}(\overline{\nabla}_{JY}JY+\overline{\nabla}_Y Y), e_i) = &\frac{\operatorname{div}\left(\rho_J\pi_L(\overline{\nabla}_{JY}JY+\overline{\nabla}_Y Y)\right)}{\rho_J}\\ &-\overline{g}\big(\pi_J(\overline{\nabla}_{JY}JY+\overline{\nabla}_Y Y), H_J\big).\end{aligned}$$

The result now follows. □

We can deduce the following important result for the second variation of the J-volume functional in the Kähler setting, which is an immediate corollary of the previous proposition.

Proposition 5.8 *Let $\iota_t : L \to L_t \subseteq M$ be compact totally real submanifolds in a Kähler manifold and let $\frac{\partial}{\partial t}\iota_t|_{t=0} = JY$ for Y tangential. Then*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\operatorname{Vol}_J(L_t)|_{t=0} = \int_L \Bigg(&\left(\frac{\operatorname{div}(\rho_J Y)}{\rho_J}\right)^2 + \overline{g}(JY, H_J)^2 - \overline{\operatorname{Ric}}(Y,Y)\\ &-\overline{g}(\pi_J(\overline{\nabla}_{JY}JY+\overline{\nabla}_Y Y), H_J)\Bigg)\operatorname{vol}_J\end{aligned}$$

We see that if L is a critical point of the J-volume functional, so $H_J = 0$, then all of the terms in the integrand are non-negative except potentially for $-\overline{\operatorname{Ric}}(Y,Y)$, so we can ensure non-negativity by imposing an ambient curvature condition. We deduce the following, which first appeared in [1].

Corollary 5.9 *Let M be a Kähler manifold with $\overline{\operatorname{Ric}} \leq 0$ (respectively, $\overline{\operatorname{Ric}} < 0$). Then the critical points of the J-volume functional are stable (respectively, strictly stable).*

Remark We can repeat our whole discussion in the almost Hermitian setting, but the appearance of torsion terms means that the second variation formula is more complicated and does not, as far as we are aware, naturally lead to a stability property for the critical points of the J-volume functional as in the Kähler case.

5.4 Convexity of the J-volume

Stability is an infinitesimal condition. We now want to show that we can obtain a much stronger result by taking into account our notion of geodesics in $\mathcal{T}$.

To start, notice that if $\overline{\mathrm{Ric}}(Y,Y) \leq 0$ then everything in the second variation formula is non-negative except potentially for $-\overline{g}(\pi_J(\overline{\nabla}_{JY}JY+\overline{\nabla}_Y Y), H_J)$. We also see that, by locally extending Y in a neighbourhood of L,

$$\overline{\nabla}_{JY}JY + \overline{\nabla}_Y Y = J(\overline{\nabla}_{JY}Y - J\overline{\nabla}_Y Y) = J[JY, Y].$$

Hence, if we deform L in a direction JY such that $[JY, Y] = 0$, which is the same as saying that the local diffeomorphisms of L generated by Y and the deformations of L generated by JY commute, then the J-volume functional is convex in the direction JY, in the sense that the second variation is non-negative. This condition is precisely guaranteed by our notion of geodesic from Lemma 2.2 so we deduce the following.

Theorem 5.10 *Let M be a Kähler manifold with $\overline{\mathrm{Ric}} \leq 0$ (respectively, $\overline{\mathrm{Ric}} < 0$). Then the J-volume functional is convex (respectively, strictly convex) in the sense of Definition 2.3.*

5.5 Critical points of the J-volume

Analogously to the Riemannian setting we will say that a totally real submanifold is *J-minimal* if it is a critical point for the J-volume, *i.e.* if $H_J = 0$.

Recall from Lemma 5.2 that the J-volume coincides with the standard volume on the subset of Lagrangian submanifolds and that the set of J-minimal Lagrangians is contained in the set of minimal Lagrangians.

In [11] we show that this result can be improved by adding assumptions on the ambient manifold. Specifically, we prove the following.

Proposition 5.11 *Assume M is almost Kähler. Then the J-volume and the standard volume coincide to first order on the set of Lagrangian submanifolds. In particular, the set of J-minimal Lagrangians coincides with the set of minimal Lagrangians.*

Assume M is Kähler-Einstein with $\overline{\mathrm{Ric}} \neq 0$. Then the set of J-minimal totally real submanifolds coincides with the set of minimal Lagrangians.

The case of Kähler Ricci-flat, in particular Calabi-Yau, manifolds is rather special. In this case, thanks to a calibration argument, J-minimal totally real submanifolds are automatically Vol_J-minimizers: we call them "special totally real submanifolds", in analogy with the well-known class of special Lagrangians, and study them in [11].

6 Abstract framework

We now introduce an abstract framework which will help us clarify and continue to analyze the structure of $\mathcal{T}$.

6.1 A canonical connection on homogeneous spaces

Let G be a finite-dimensional Lie group. Let L and R denote the left and right multiplication operators, Ad the adjoint action of G on G and ad its differential, inducing an action of G on T_eG. Let $\mathfrak{g}$ denote the Lie algebra of G, *i.e.* the space of L-invariant vector fields. In the course of this section it will be useful to emphasize the distinction between T_eG and $\mathfrak{g}$, using the notation

$$T_eG \to \mathfrak{g}, \quad X \mapsto \tilde{X}$$

to refer to the isomorphism induced by L.

Given $X \in T_eG$, consider the 1-dimensional subgroup of diffeomorphisms of G defined by the flow ϕ_t of $\tilde{X}$:

$$\frac{d}{dt}\phi_t = \tilde{X}_{|\phi_t}, \quad \phi_0 = \mathrm{Id}. \tag{11}$$

The above-mentioned isomorphisms $(L_g)_* : T_eG \to T_gG$ allow us to identify each tangent space with T_eG, thus inducing a canonical way to differentiate vector fields. This yields a connection on TG, generally known as the *canonical L-invariant connection*. The parallel vector fields are the elements of $\mathfrak{g}$, so the flowlines of ϕ_t are precisely the geodesics of this connection. In particular, the geodesics through e are the 1-parameter subgroups $\exp(tX) := \phi_t(e)$. The L-invariance of the connection implies that L preserves the geodesics. This is reflected in the fact that, for any $g \in G$, $L_g\phi_t$ coincides with the flowline passing through g.

Now fix a closed subgroup H. Assume there exists a decomposition

$$T_eG = T_eH \oplus M, \quad \text{for some } ad_H\text{-invariant subspace } M.$$

Let $\mathfrak{h}$, $\mathfrak{m}$ denote the corresponding L-invariant distributions on G so that $TG = \mathfrak{h} \oplus \mathfrak{m}$. Consider the projection $\pi : G \to G/H$, viewed as an H-principal fibre bundle. Notice that $\mathfrak{h}$ is tangent to the R_H-action: choosing $g \in G$ and a 1-parameter subgroup h_t in H, we see that

$$\frac{d}{dt}R_{h_t}g_{|t=0} = \frac{d}{dt}gh_{t|t=0} = (L_g)_*\frac{d}{dt}h_{t|t=0} \in \mathfrak{h}_{|g}.$$

This is a manifestation of the general fact that L-invariant vector fields coincide with the fundamental vector fields of the R-action.

We can also check that $\mathfrak{m}$ is R_H-invariant: choosing $X \in M$ so that $(L_g)_*X \in \mathfrak{m}_{|g}$, we see that

$$(R_h)_*(L_g)_*X = (L_g)_*(R_h)_*X = (L_g)_*(L_h)_*Y = (L_{gh})_*Y \in \mathfrak{m}_{|gh},$$

where we use the fact that M is ad_H-invariant so $(L_{h^{-1}})_*(R_h)_*X = Y$, for some $Y \in M$.

The splitting $TG = \mathfrak{h} \oplus \mathfrak{m}$ thus defines a connection on the principal fibre bundle, and induced connections on all associated bundles of the form $G \times_\rho V$, where ρ is a G-action on the vector space V.

The following result shows that one of these bundles is particularly relevant to the geometry of G/H.

Proposition 6.1 *There is an isomorphism*

$$G \times_{ad_H} M \to T(G/H), \quad [g, X] \mapsto \pi_*(L_g)_* X.$$

Thus G/H has a canonical connection induced from the connection $\mathfrak{m}$ on the principal bundle G. Geodesics in G/H are of the form $\pi(g_t)$, where g_t is a horizontal curve in G satisfying the equation

$$\frac{d}{dt} g_t = (L_{g_t})_* X \quad (\textit{equivalently, } \frac{d}{dt} g_t = \tilde{X}_{|g_t})$$

for some fixed $X \in M$. In other words, geodesics in G/H are the projections of geodesics in G defined by the canonical L-invariant connection, with initial direction in M.

6.2 Geometry of complexified Lie groups

Let G^c be a complexified Lie group, *i.e.* a complex Lie group with Lie algebra isomorphic to $\mathfrak{g} \otimes \mathbb{C}$. We want to study the homogeneous space G^c/G.

The maps L and R are holomorphic, so each operator $ad_g : T_e G^c \to T_e G^c$ commutes with the complex structure J on G^c. This implies that $M := J(T_e G)$ is ad_G-invariant, so we can apply the above theory using the splitting $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$. According to Proposition 6.1, there is an isomorphism

$$G^c \times_{ad_G} (JT_e G) \to T(G^c/G), \quad [g, JX] \to \pi_*(L_g)_*(JX) = \pi_* J(L_g)_* X. \tag{12}$$

It follows that G^c/G has a canonical connection, whose geodesics are the projection of curves g_t in G^c satisfying the equation

$$\frac{d}{dt} g_t = J(L_{g_t})_* X,$$

for some $X \in T_e G$. This is an ordinary differential equation on the space G^c. If G^c is infinite-dimensional there may be no solutions; however, if a solution does exist for one given initial point, it will exist for any initial point because the equation is L-invariant. In particular, the solution corresponding to the initial point $e \in G^c$ is the 1-parameter subgroup $\exp(tJX) \subset G^c$.

We can try to also integrate the vector field X, obtaining a 1-parameter subgroup $\exp(sX) \subset G$. Assume these subgroups exist. Consider the real 2-dimensional distribution in TG^c generated by X and JX. Since the Lie bracket commutes with J we see that $[X, JX] = 0$, so the distribution is integrable and our integrations yield a 1-dimensional complex abelian Lie subgroup of G^c, spanned by $\exp(sX)$, $\exp(tJX)$. Abstractly, it is the complexification of the Lie group $\exp(sX)$; it is isomorphic to $\mathbb{S}^1 \times \mathbb{R}$ or to $\mathbb{C}$ depending on whether $\exp(sX)$ is compact or not.

Summarizing, the collection of geodesics in G^c/G is equivalent (through projection and L-invariance) to the collection of real 1-parameter subgroups in G^c generated by JX, or to the collection of complex 1-parameter subgroups in G^c generated by X, for $X \in T_e G$.

The above applies also to the boundary value problem for geodesics in G^c/G: any geodesic $\gamma(t)$, for $t \in [a,b]$, interpolating between two points in G^c/G lifts to a holomorphic map $\Sigma \to G^c$, where $\Sigma := \mathbb{S}^1 \times [a,b]$ or $\Sigma := \mathbb{R} \times [a,b]$, with prescribed boundary values. More generally one can study the existence of holomorphic maps $\Sigma \to G^c$ with prescribed boundary values, where Σ is any given Riemann surface with boundary.

Notation. From now on we will often relax the distinction between T_eG and $\mathfrak{g}$, and the corresponding distinction between X and $\tilde{X}$.

Definition 6.2 *A function $f : G^c/G \to \mathbb{R}$ is strictly convex if it is strictly convex when restricted to all geodesics in G^c/G. Equivalently, if the lifted function $F := \pi^* f : G^c \to \mathbb{R}$ satisfies*

$$JX(JX(F)) = \frac{d^2}{dt^2}(F \circ g_t) > 0,$$

for all geodesics g_t in G^c with velocity JX, for some $X \in T_eG$.

Proposition 6.3 *Any strictly convex function $f : G^c/G \to \mathbb{R}$ lifts to a Kähler potential $F := \pi^* f$ on G^c.*

Proof: Consider the 2-form $\omega_f := i\partial\bar{\partial}F = \frac{1}{2}dd^cF$ defined on G^c. By construction it is of type $(1,1)$. We need to show that it is positive, *i.e.* that the symmetric tensor $\omega_f(\cdot, J\cdot)$ is positive definite. This is a pointwise statement which must be tested on every vector in $T_gG^c = (\mathfrak{g} + i\mathfrak{g})_{|g}$, for all $g \in G^c$. Equivalently, it suffices to prove that ω_f is positive when restricted to any complex line. Since $i\mathfrak{g}$ is totally real of maximal dimension, it must intersect the line so we may assume our line is generated by a vector X in $i\mathfrak{g}$. For our computation it is then sufficient to consider the restriction of F to the submanifold of G^c obtained by integrating the vector fields X, JX. We now see that our problem corresponds to the $n = 1$ case of the following fact: given $f : \mathbb{R}^n \to \mathbb{R}$ and $F := \pi^* f : \mathbb{R}^{2n} \to \mathbb{R}$, if f is strictly convex then $i\partial\bar{\partial}F$ is positive. Indeed, it is simple to compute that

$$i\partial\bar{\partial}F = i\frac{\partial^2 F}{\partial z_i \partial \overline{z_j}} dz^i \wedge d\overline{z^j} = 2\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx^i \wedge dy^j,$$
$$i\partial\bar{\partial}F((X,Y),(-Y,X)) = 2\frac{\partial^2 f}{\partial x_i \partial x_j} x^i x^j + 2\frac{\partial^2 f}{\partial x_i \partial x_j} y^i y^j.$$

Trivially, ω_f is closed so the result follows. □

The above proposition shows that any strictly convex function f on G^c/G defines a Kähler structure ω_f on G^c. Recall that G acts holomorphically on G^c. It also preserves the Kähler potential, so it preserves ω_f.

Proposition 6.4 *The action of G on G^c, endowed with a Kähler structure ω_f, is Hamiltonian with moment map*

$$\mu_f := -\frac{1}{2}d^cF = \frac{1}{2}dF \circ J : G^c \to \mathfrak{g}^*.$$

In particular, the zero set of the moment map is the inverse image, under π, of the critical set of f. Since f is strictly convex the critical set is either empty or a unique point, so the zero set is either empty or a unique G-orbit in G^c.

Proof: For any $X \in \mathfrak{g}$, duality with $\mathfrak{g}^*$ defines a function

$$\mu_f \cdot X := -\frac{1}{2} i_X d^c F : G^c \to \mathbb{R}.$$

We want to show that X is the Hamiltonian vector field associated to this function, *i.e.* $d(-i_X d^c F) = 2 i_X \omega_f$.

Using Cartan's formula for the Lie derivative of the 1-form $d^c F$, we see

$$d(-i_X d^c F) = -\mathcal{L}_X d^c F + i_X dd^c F = \mathcal{L}_X (dF \circ J) + 2 i_X \omega_f.$$

The first term vanishes because both F and J are preserved by the action of G.

To conclude, notice that $g \in G^c$ lies in the zero set of the moment map if and only if $(dF \circ J)_{|g}(X) = dF_{|g}(JX) = 0$ for all $X \in \mathfrak{g}$. Since F is G-invariant this is equivalent to $dF_{|g} = 0$, thus $df_{|\pi(g)} = 0$. □

6.3 Existence of critical points via a stability condition

The interpretation of critical points of $f : G^c/G \to \mathbb{R}$ as zeros of a moment map is geometrically interesting but in itself does not bring us closer to understanding whether, in any specific situation, such points exist. If however we can embed G^c with its given structures into a larger Kähler manifold M, we can sometimes apply the following general framework for studying this existence problem.

Let $(M, \overline{g}, J, \overline{\omega})$ be a Kähler manifold endowed with a G-action preserving $\overline{g}$ and J. To simplify matters we assume this action is free. Let us assume that the complexified group G^c also acts on M, preserving J: this gives a family of G^c-orbits in M. As a first step, we are interested in finding situations in which each such orbit $\mathcal{O}$ admits a canonical function F such that $\overline{\omega}_{|\mathcal{O}} = i\partial\bar{\partial}F$ as in Proposition 6.3. An example of this is as follows.

Assume M is polarized, *i.e.* there exists a holomorphic line bundle L over M with a Hermitian metric such that the corresponding Chern connection has curvature $\Theta = i\overline{\omega}$. Recall the standard formula for Θ in terms of a local holomorphic section: $\Theta = \bar{\partial}\partial \log|\sigma|^2$. It follows that $\bar{\omega} = i\partial\bar{\partial}\log|\sigma|^2$. If the action of G is Hamiltonian, the moment map yields a canonical way to lift the infinitesimal action of G to the total space of L, cf. [5, Section 6.5] for details. Let us assume this integrates to an action of G^c. Any point in L then generates a G^c-orbit $\tilde{\mathcal{O}}$ which projects to a G^c-orbit $\mathcal{O}$ in M. Let us think of $\tilde{\mathcal{O}}$ as the graph of a non-vanishing holomorphic section $\sigma : \mathcal{O} \subset M \to L$. Consider the function $F := \log|\sigma|^2 : \mathcal{O} \to \mathbb{R}$. By construction G preserves the Hermitian metric so $F = \pi^* f$, for some $f : G^c/G \simeq \mathcal{O}/G \to \mathbb{R}$. It turns out that f is strictly convex in our sense; the formula for ω corresponds exactly to the situation of Proposition 6.3.

In the above situation we say that an orbit $\mathcal{O}$ is *stable* if f is proper, so that it admits a critical point. Let M^s denote the set of points in M whose corresponding G^c-orbits are stable. According to Proposition 6.4 there is a 1:1 mapping

$$M^s/G^c \simeq \mu^{-1}(0)/G.$$

The key point is that, in specific situations, stability of a given orbit can sometimes be tested using purely holomorphic information on M and the G^c-action. We thus get a correspondence between holomorphic and symplectic data on M, addressing the existence of critical points of the functions f.

Summarizing: if we manage to embed our given complexified group G^c, endowed with the structure ω_f defined by a strictly convex function $f : G^c/G \to \mathbb{R}$, into some Kähler M with a G^c-action so that it coincides with one of these orbits, then we can hope to test the existence of critical points of f by verifying some type of "stability condition" on that orbit.

6.4 Extension to infinitesimal complexifications

Finite-dimensional examples of stability and its relation to existence problems are a classical topic of Algebraic Geometry, related to Geometric Invariant Theory and the Kempf–Ness theorem.

Gauge theory provided the first context in which this abstract framework arose in an infinite-dimensional setting: this is related to the Hitchin–Kobayashi conjecture concerning the existence of Hermitian–Einstein connections on a given Hermitian vector bundle E over a Kähler manifold, cf. [5] for details. In this case G is the group of unitary transformations of E, and its complexification G^c is the group of automorphisms of E.

In general however when G is infinite-dimensional there does not exist a complexification G^c, cf. [10]. The above theory can thus not be applied as it stands. For this reason Donaldson [4] introduced a slightly weaker notion of "infinitesimal complexification" of a group G. Within this framework we can recover all the above results, as follows.

Definition 6.5 *Let Z be a smooth manifold. Assume there exists a vector space V and an injection*

$$V \to \Lambda^0(TZ), \quad X \mapsto \tilde{X}$$

such that the vector fields $\tilde{X}$ define a parallelization of TZ, thus $TZ \simeq Z \times V$. Assume further that the space of vector fields $\tilde{X}$ is closed under the Lie bracket on Z. We then get an induced Lie bracket on V such that $\widetilde{[X,Y]} = [\tilde{X}, \tilde{Y}]$.

We say that the above data defines an infinitesimal Lie group Z with Lie algebra V.

Definition 6.6 *Let (Z, J) be a complex manifold. Assume there exists a Lie group G acting freely on the right on Z and preserving J. Given $X \in \mathfrak{g}$, let $\tilde{X}$ be*

the corresponding fundamental vector field: specifically, if X is the infinitesimal deformation of the 1-parameter subgroup g_t then

$$\tilde{X}_{|\zeta} := \frac{d}{dt}(\zeta \cdot g_t)_{|t=0}.$$

This defines an injection $\mathfrak{g} \to \Lambda^0(TZ)$ preserving the corresponding Lie brackets.

Consider the extended map

$$\mathfrak{g} \otimes \mathbb{C} \to \Lambda^0(TZ), \quad X + iY \mapsto \tilde{X} + J\tilde{Y}. \tag{13}$$

Assume it is injective and provides a parallelization of TZ. The fact that G preserves J implies that $\mathcal{L}_{\tilde{X}} J = 0$, i.e. $[\tilde{X}, JY] = J[\tilde{X}, Y]$ for all $Y \in \Lambda^0(TZ)$. The fact that the Nijenhuis tensor vanishes implies that also $\mathcal{L}_{J\tilde{X}} J = 0$. It follows that the image of the map (13) is closed under the Lie bracket on Z and that this Lie bracket is J-linear, i.e. the image is a complex Lie algebra. Thus (13) defines a complex Lie algebra isomorphism onto its image.

We then say that Z is an infinitesimal complexification of G.

Given Z and G as above, we can view $\pi : Z \to Z/G$ as a principal G-bundle. The fundamental vector fields $\tilde{X}$ define the "vertical space", *i.e.* the kernel of π_*. The space of fields $J\tilde{X}$ defines a complementary distribution, which is G-invariant because G preserves J. In other words, the splitting

$$TZ \simeq \mathfrak{g} \oplus i\mathfrak{g}$$

defines a connection on Z, thus on all associated bundles.

A priori there is no adjoint action of G on $i\mathfrak{g}$, because there is no actual group G^c inducing it. We can however define an *ad hoc* action using the adjoint action of G on $\mathfrak{g}$, as follows:

$$ad_G : G \to GL(i\mathfrak{g}), \quad ad_g(iX) := i\, ad_g(X). \tag{14}$$

This allows us to define the associated bundle $Z \times_{ad_G} (i\mathfrak{g})$.

Proposition 6.7 *There is an isomorphism*

$$Z \times_{ad_G} (i\mathfrak{g}) \to T(Z/G), \quad [\zeta, X] \mapsto \pi_*(J\tilde{X}_{|\zeta}).$$

Proof: The main issue is to check that the map is well-defined, *i.e.* that the images of $[\zeta \cdot g, ad_{g^{-1}}X]$ and of $[\zeta, X]$ coincide. It suffices to prove that $\widetilde{ad_{g^{-1}}X}_{|\zeta \cdot g} = (\tilde{X}_{|\zeta}) \cdot g$, which is a simple computation. □

Remark When $Z = G^c$ is a standard complexification, this construction coincides with the previous one in (12) because $(L_g)_* X$ is the fundamental vector field $\tilde{X}$ of the right action of G on G^c.

As above, it follows that Z/G has a canonical connection whose geodesics are the projection of curves ζ_t in Z satisfying the equation

$$\frac{d}{dt}\zeta_t = J\tilde{X}_{|\zeta_t},$$

for some $X \in T_eG$. As in Section 6.2 this is an ordinary differential equation: solving it corresponds to integrating the vector field $\zeta_t \mapsto J\tilde{X}_{|\zeta_t}$ in Z. This problem is G-invariant, but in this context there is no notion of G^c-invariance. We can complexify geodesics by combining them with solutions to the equation $\frac{d}{ds}\zeta_s = \tilde{X}_{|\zeta_s}$, thus obtaining holomorphic curves in G^c.

The analogues of Propositions 6.3 and 6.4 continue to hold in this context.

7 Kähler potentials and cscK metrics

In Section 6.4 we mentioned that the ideas of Section 6 can be usefully applied to gauge theory. A second geometric setting in which this abstract framework proves itself useful is the search for constant scalar curvature Kähler (cscK) metrics on a complex manifold (M, J) within a given Kähler class $[\omega_0]$. In this case the appropriate Lie group G does not admit a formal complexification, so it is necessary to work with the infinitesimal complexifications described in Section 6.4. The goal of this section is to provide an overview of this problem in such a way as to emphasize analogies with the main topic of this paper: the space of totally real submanifolds, the existence of geodesics and the search for critical points of the J-volume.

Let M be a compact manifold. The space $\mathrm{Diff}(M)$ can be given the structure of an infinite-dimensional Lie group with Lie algebra $\mathcal{X} := \Lambda^0(TM)$. As for any Lie group, the tangent space $T_\zeta \mathrm{Diff}(M)$ is spanned by vectors of the form $(L_\zeta)_*X$, for $X \in \mathcal{X}$; alternatively, by $(R_\zeta)_*Y = Y \circ \zeta$, for $Y \in \mathcal{X}$. In words: tangent vectors at ζ are sections of the pullback bundle ζ^*TM. If $(L_\zeta)_*X = (R_\zeta)_*Y$, then X and Y are related by the *ad*-action, which in this context coincides with the standard "pushforward" action: $ad_\zeta(X) = d\zeta_{|\zeta^{-1}}(X_{|\zeta^{-1}})$.

If M has a complex structure J, $\mathrm{Diff}(M)$ receives an induced complex structure $J(X_{|\zeta}) := (JX)_{|\zeta}$. The smooth structure on $\mathrm{Diff}(M)$ is defined so that the corresponding Lie bracket can be calculated in terms of the Lie bracket on $\mathcal{X}$: it follows that the Nijenhuis tensor of J vanishes on $\mathrm{Diff}(M)$ if this is true on M. The right action R preserves J but L does not, so $\mathrm{Diff}(M)$ is a complex manifold but not a complex Lie group.

Now assume (M, J, ω_0) is Kähler. Consider the space $\mathcal{H}$ of Kähler structures in the cohomology class defined by ω_0. This is a convex subspace of the space of 2-forms on M. According to the $\partial\bar{\partial}$-lemma, any such ω can be written as

$$\omega = \omega_f := \omega_0 + i\partial\bar{\partial}f,$$

for some $f \in C^\infty(M)$. The potential f is well-defined only up to a constant; we can choose a canonical representative for f using a normalization functional $I : C^\infty(M) \to \mathbb{R}$ introduced by Bando and Mabuchi, with the following properties.

- There is a 1:1 identification

$$I^{-1}(0) \simeq \mathcal{H}, \quad f \mapsto \omega_f. \tag{15}$$

- The tangent space at $f \in I^{-1}(0)$ is

$$T_f I^{-1}(0) = \{h \in C^\infty(M) : \int_M h \,\mathrm{vol}_{\omega_f} = 0\}.$$

We will alternatively denote this space $C^\infty_{\omega_f}(M)$.

We refer to [4] for details. Using this identification, for any $f \in \mathcal{H}$ we define

$$\mathcal{Q}_f := \{\zeta \in \mathrm{Diff}(M) : \zeta^*\omega_f = \omega_0\}.$$

Let $\mathcal{Q} \subset \mathrm{Diff}(M)$ denote the union of all such $\mathcal{Q}_f$. Consider the right action of the subgroup of Hamiltonian diffeomorphisms $G := \mathrm{Ham}(M, \omega_0)$ on $\mathrm{Diff}(M)$. Each $\mathcal{Q}_f$ is an orbit of this action so $\pi : \mathcal{Q} \to \mathcal{H}$ is a principal G-bundle.

For any symplectic structure ω we will use the notation $\mathrm{Ham}(\mathcal{X}, \omega)$ to denote the Lie algebra of $\mathrm{Ham}(M, \omega)$. Its elements are the vector fields X^ω_h satisfying the equation $dh = \omega(X^\omega_h, \cdot)$, for some function $h : M \to \mathbb{R}$. In the Kähler setting it follows that $X^\omega_h = -J\nabla^\omega h$, where ∇^ω is the gradient operator defined by the induced metric $g := \omega(\cdot, J\cdot)$. Once again we can choose h uniquely by normalizing it so that it belongs to $C^\infty_\omega(M)$. It follows that we can identify the Lie algebra $\mathrm{Ham}(M, \omega)$ with the Lie algebra $C^\infty_\omega(M)$, endowed with the natural Poisson bracket on functions (up to sign).

Lemma 7.1 *The adjoint action of* $\mathrm{Diff}(M)$ *on* $\mathcal{X}$ *satisfies*

$$ad_\zeta(X^{\omega_0}_h) = X^{\omega_f}_{h\circ\zeta^{-1}},$$

for all $\zeta \in \mathcal{Q}_f$. *Furthermore, if* $h \in C^\infty_{\omega_0}(M)$ *then* $h \circ \zeta^{-1} \in C^\infty_{\omega_f}(M)$.

Proof: One can check that the two vector fields coincide when contracted with ω_f. The normalization property of $h \circ \zeta^{-1}$ is a straightforward computation. □

The vertical space of the fibration π at a point $\zeta \in \mathcal{Q}_f$ is the subspace $(L_\zeta)_* \mathrm{Ham}(\mathcal{X}, \omega_0)$ of $T_\zeta \mathcal{Q}$. It follows from Lemma 7.1 that $(L_\zeta)_* \mathrm{Ham}(\mathcal{X}, \omega_0) = (R_\zeta)_* \mathrm{Ham}(\mathcal{X}, \omega_f) = \mathrm{Ham}(\mathcal{X}, \omega_f)_{|\zeta}$. One can check that there is a splitting

$$T_\zeta \mathcal{Q} = \mathrm{Ham}(\mathcal{X}, \omega_f)_{|\zeta} \oplus J\,\mathrm{Ham}(\mathcal{X}, \omega_f)_{|\zeta}.$$

It follows that $\mathcal{Q}$ is an infinitesimal complexification of $\mathrm{Ham}(M, \omega_0)$ in the sense of Definition 6.6. Equivalently, $T\mathcal{Q}$ is parallelized by the map

$$C^\infty_{\omega_0}(M) \otimes \mathbb{C} \to \Lambda^0(T\mathcal{Q}), \quad h + ik \mapsto \Big(\zeta \mapsto \big(X^{\omega_f}_{h\circ\zeta^{-1}} + JX^{\omega_f}_{k\circ\zeta^{-1}}\big)_{|\zeta}\Big).$$

As a first consequence of this formalism, we learn that $\mathcal{H} = \mathcal{Q}/G$ has a canonical connection. As above, the geodesics are the curves $f_t \subset \mathcal{H}$ obtained by projection of solutions $\zeta_t \subset \mathcal{Q}$ to the equation

$$\frac{d}{dt}\zeta_t = J\left(X^{\omega_{f_t}}_{h\circ\zeta_t^{-1}}\right)_{|\zeta_t}, \tag{16}$$

where h is a time-independent function in $C^\infty_{\omega_0}(M)$ and $\zeta_t \in \mathcal{Q}_{f_t}$. As mentioned in Section 6.4 we can also integrate the equation $\frac{d}{ds}\zeta_s = \tilde{X}_{|\zeta_s} := (X^{\omega_f}_{h\circ\zeta_s^{-1}})_{|\zeta_s}$, thus obtaining holomorphic curves in $\mathcal{Q}$: these are the *complexified Hamiltonian flows* mentioned in [4].

According to our identifications, the right-hand side of (16) projects to the tangent vector $h \circ \zeta_t^{-1} \in T_{f_t}\mathcal{H}$, so the projected equation is

$$\frac{d}{dt}f_t = h \circ \zeta_t^{-1}, \tag{17}$$

for some t-independent h. We can incorporate this condition into the equation by noticing that $h = \frac{d}{dt}f_t \circ \zeta_t = \dot{f}_t \circ \zeta_t$ (notice the change in notation). The geodesic equation on $\mathcal{H}$ is thus

$$\dot{h} = \ddot{f}_{t|\zeta_t} + d\dot{f}_{t|\zeta_t}(\dot{\zeta}_t) = 0.$$

We can express the right-hand side of (16) as the gradient $\nabla^{\omega_{f_t}}(\dot{f}_t)$. We thus arrive at the final expression for geodesics on $\mathcal{H}$:

$$\ddot{f}_{t|\zeta_t} + |\nabla^t \dot{f}_{t|\zeta_t}|^2_t = 0, \tag{18}$$

where we simplify notation by using ∇^t, $|\cdot|_t$ to indicate that we are using the metric induced by ω_{f_t}.

Remark It is useful to compare equations (17), (18): the former is first order, expressing the fact that $\dot{f}_t$ coincides with a parallel vector field; the latter is second order and uses the fact that $C^\infty(M)$ is a vector space, thus has a natural connection. In other words, (18) expresses the geodesics of the canonical connection on $\mathcal{Q}/G$ in terms of the natural connection on $C^\infty(M)$.

We now turn to the problem of finding a cscK metric in $\mathcal{H}$. It turns out that there exists a functional $f : \mathcal{H} \to \mathbb{R}$, due to Mabuchi, with the following properties:

- f is convex with respect to the geodesics defined above;
- the critical points of f are precisely the potentials of cscK metrics in $\mathcal{H}$.

Now consider the space $\mathcal{J}$ of integrable complex structures on M which are compatible with ω_0. Let $G := \mathrm{Ham}(M, \omega_0)$ act on $\mathcal{J}$ by pullback. It can be shown that $\mathcal{J}$ has a canonical Kähler structure which is preserved by the action of G. Furthermore, it is possible to embed $\mathcal{Q}$, together with the Kähler structure defined by f according to Proposition 6.3, into $\mathcal{J}$: this is described *e.g.* in [7, Chapter 9], so we do not review it here. As in Section 6.2, this embedding provides strong geometric motivation for the Yau–Tian–Donaldson conjecture concerning stability conditions related to the existence of a cscK metric in $\mathcal{H}$.

8 Complexified diffeomorphism groups

Let (M, J) be a complex manifold. In Section 7 we argued that $\mathrm{Diff}(M)$ inherits a complex structure which is formally integrable. The same construction applies to the space of immersions $\mathcal{I}$ of L into M; the space $\mathcal{P}$ is then an open subset of $\mathcal{I}$, so it is formally an infinite-dimensional complex manifold. The action of $\mathrm{Diff}(L)$ by reparametrization preserves the complex structure. We can thus reformulate the material of Section 2 in terms of the formalism of Section 6. We conclude the following.

- $\mathcal{P}$ is an infinitesimal complexification of $\mathrm{Diff}(L)$.
- Proposition 6.7 applies to $\mathcal{P}$, proving that the tangent space $T(\mathcal{T})$ can be identified with the adjoint bundle associated to $\mathcal{P}$.
- The connection and geodesics defined in Section 3 coincide with those defined in Section 6.4.
- When M is Kähler and $\overline{\mathrm{Ric}}(M) < 0$, Proposition 6.3 shows that the functional Vol_J defines a Kahler structure ω_J on $\mathcal{P}$. Proposition 6.4 also applies, showing that the critical points of Vol_J can be interpreted as the zero set of a moment map.

Remark A theorem of Bruhat and Whitney [16] shows that any real analytic L can be "complexified", *i.e.* embedded as a totally real submanifold into an appropriate complex manifold (M, J). It thus defines a space $\mathcal{P}$. It follows that the corresponding group $\mathrm{Diff}(L)$ admits an infinitesimal complexification even though it may not admit a genuine complexification [10].

A special case of the above occurs when M is negative Kähler–Einstein: in this case, using Proposition 5.11, we obtain a reformulation of minimal Lagrangians in terms of the zero set of a moment map. The analogies with the theory of cscK metrics and Hermitian–Einstein connections lead to the following question, which seems worthy of further pursuit.

Question Can the existence of minimal Lagrangian submanifolds in negative KE manifolds be related to a stability condition concerning $(M, \overline{g}, J, \bar{\omega})$ and the chosen homotopy class $\mathcal{T}$?

The corresponding uniqueness question should be related to the convexity of the J-volume functional, but requires a better understanding of the existence question for geodesics.

It would also be worth comparing these ideas to the stability conditions and uniqueness results for minimal Lagrangians in negative KE manifolds conjectured in [8].

References

[1] Vincent Borrelli, *Maslov form and J-volume of totally real immersions*, J. Geom. Phys. **25** (1998), no. 3-4, 271–290.

[2] Bang-yen Chen, *Geometry of submanifolds and its applications*, Science University of Tokyo, Tokyo, 1981.

[3] John B. Conway, *Functions of one complex variable. II*, Graduate Texts in Mathematics, vol. 159, Springer-Verlag, New York, 1995.

[4] Simon K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, Northern California Symplectic Geometry Seminar, Amer. Math. Soc. Transl. Ser. 2, vol. 196, Amer. Math. Soc., Providence, RI, 1999, pp. 13–33.

[5] Simon K. Donaldson and Peter B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1990, Oxford Science Publications.

[6] Peter L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970.

[7] Paul Gauduchon, *Calabi's extremal Kähler metrics: An elementary introduction*, lecture notes.

[8] Dominic Joyce, *Conjectures on Bridgeland stability for Fukaya categories of Calabi-Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow*, EMS Surv. Math. Sci. **2** (2015), no. 1, 1–62.

[9] Andreas Kriegl and Peter W. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence, RI, 1997.

[10] László Lempert, *The problem of complexifying a Lie group*, Multidimensional complex analysis and partial differential equations (São Carlos, 1995), Contemp. Math., vol. 205, Amer. Math. Soc., Providence, RI, 1997, pp. 169–176.

[11] Jason D. Lotay and Tommaso Pacini, *From Lagrangian to totally real geometry: coupled flows and calibrations*, available on www.arxiv.org (2014).

[12] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2004.

[13] Walter Rudin, *Functional analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[14] Michael Spivak, *A comprehensive introduction to differential geometry. Vol. V*, second ed., Publish or Perish, Inc., Wilmington, Del., 1979.

[15] François Trèves, *Basic linear partial differential equations*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Pure and Applied Mathematics, Vol. 62.

[16] H. Whitney and F. Bruhat, *Quelques propriétés fondamentales des ensembles analytiques-réels*, Comment. Math. Helv. **33** (1959), 132–160.